Unification of Gravity and Electromagnetism
Mohammed A. El-Lakany
E-mail: mohammed_ellakani@yahoo.com
Physics Department, Faculty of Science, Cairo University, Giza, Egypt

Abstract.
Discovery of the link between gravity and electromagnetism lead to good understanding of physics; Gravity and electromagnetism are two sides of the same coin; this is the clue of this unification; the two sides are representing by two mathematical structures, symmetric represents gravity and antisymmetric represent electromagnetism. Einstein’s gravitational field equation represents the symmetric mathematical structure and we will use stress energy tensor definition for each part in electrodynamics Lagrangian to construct the antisymmetric mathematical structure; electrodynamics Lagrangian is three parts, for electromagnetic field, Dirac field and third part for interaction. Symmetric and antisymmetric gravitational field equations are two sides of the same Lagrangian.
Keyword: Gravity; Electromagnetism; General theory of Relativity; Quantum field theory; Nuclear and Particle physics; Astrophysics and cosmology.

1. Gravity and electromagnetism are two sides of the same coin.
Gravitational objects have similarly quantum properties for elementary particles; the angular momentum for the sun is given by $J_{\text{sun}} = M_{\text{sun}} \omega_{\text{sun}} R_{\text{sun}}^2 \approx 10^{40}$ ergs.s ; for solar system is $J_{\text{solsys}} = M_{\text{solsys}} \omega_{\text{solsys}} R_{\text{solsys}}^2 \approx 10^{52}$ ergs.s. In the case of a galaxy the angular momentum is given by $J_{\text{gal}} = M_{\text{gal}} \omega_{\text{gal}} R_{\text{gal}}^2$ where $M_{\text{gal}} = 10^{45}$ g; $R_{\text{gal}}^2 = 10^{47}$ cm$^2$; $\omega_{\text{gal}} = 2 \times 10^{-18}$ HZ and the value of angular momentum is $J_{\text{gal}} \approx 10^{74}$ ergs.s. Similarly for cluster of galaxies, the angular momentum is given by $J_{\text{Clust}} = M_{\text{Clust}} \omega_{\text{Clust}} R_{\text{Clust}}^2 \approx 10^{110}$ h in Hubble scale and for the universe $J_{\text{univ}} \approx 10^{120}$ h ., spin density ($\sigma = \text{spin/volume}$) is the same for a wide range from elementary particles to gravitational objects. For an electron; the spin density is given by $\sigma_e = \frac{0.5h}{\frac{4}{3} \pi r_e^3} \sim 10^9 \text{ergs.s/cc}$. For proton $\sigma_p \sim 10^9 \text{ergs.s/cc}$ also for the solar system we have $\sigma_{\text{solsys}} \sim 10^9 \text{ergs.s/cc}$; for a galaxy $\sigma_{\text{gal}} = \frac{10^{100} h}{\frac{4}{3} \pi R_{\text{gal}}^3} \sim 10^9 \text{ergs.s/cc}$, spin density for Universe $\sigma_{\text{univ}} = \frac{10^{120} h}{\frac{4}{3} \pi R_H^3} \sim 10^9 \text{ergs.s/cc}$ [32]. Not only this, but also magnetic fields seem to be everywhere that we can look in the universe [2].

Observations indicate that magnetic fields are associated with gravitational objects in the universe; Magnetic fields are observed to be of the order of $10^{12}$ G in neutron stars, $10^3$ G in solar type Stars. Magnetic fields of order a few μG also have been detected in radio galaxies [3]. Gravitational objects are magnetic dipoles; electromagnetism not tied only to charged particles, but the planets, stars, galaxies and clusters. gravity and electromagnetism are two sides of the same coin.

2. Symmetric and antisymmetric mathematical structures
Unification of gravity and electromagnetism has been pursued by many scientists, like Weyl, Eddington, Einstein, Infeld, Born and Schrodinger. Weyl’s initiated this unification; Eddington considered connection as the central concept then decomposed its Ricci tensor to symmetric Ricci tensor \( R_{\mu\nu} \) represents gravity and antisymmetric Ricci tensor \( R_{\nu\sigma} \) represent electromagnetism.

Infeld - Born followed the path of Eddington and deduced the Lagrangian \( \mathcal{L}_{GR} = \sqrt{-\det(g_{\mu\nu} + F_{\nu\sigma})} - \sqrt{-g} \), they considered the asymmetric metric \( g_{(\mu\nu)} = g_{\mu\nu} + F_{\nu\sigma} \), its symmetric part \( (g_{\mu\nu}) \) represents gravity and antisymmetric part \( (F_{\nu\sigma}) \) represent electromagnetism, \((g)\) is the determinant of the symmetric metric tensor \( (g_{\mu\nu}) \) [42]. Schrodinger generalized Eddington Lagrangian to new form containing the cosmological constant \( (\Lambda) \) [10]; despite the failure of these previous attempts, they in its entirety refers to something cannot be neglected is that gravity and electromagnetism should be represented by two mathematical structures.

3. Curvature tensor

Riemann tensor in terms of Christoffel’s 3-index symbols defined by
\[
R^\delta_{\mu\nu\sigma} = \Gamma^\delta_{\mu\sigma} \Gamma^\mu_{\lambda\nu} - \Gamma^\delta_{\mu\nu} \Gamma^\mu_{\lambda\sigma} + \Gamma^\delta_{\mu\sigma\nu} - \Gamma^\delta_{\nu\mu\sigma},
\]
(1)

Riemann Christoffel tensor is of rank four, contravariant in \( \delta \) and covariant in \( \mu, \nu, \) and \( \sigma \), and also
\[
R^\delta_{\mu\nu\sigma} = 0
\]
(2)

Is the necessary condition for the validity of the special theory of Relativity and for the absence of permanent gravitational field or are the necessary and sufficient condition that the space time is flat[17].

Lowering the last index in the Riemann Christoffel tensor with the symmetric metric tensor, the lowered tensor \( R_{\mu\nu\sigma\epsilon} = R^\delta_{\mu\nu\sigma} g_{\delta\epsilon} \) is symmetric under Interchanging of the first and last pair of indices and antisymmetric in \( \mu, \epsilon \) and in \( \nu, \sigma \). Symmetric and antisymmetric Ricci tensors can be written in the form;
\[
R^\delta_{\mu\nu} = R^\delta_{\nu\mu} = \Gamma^\delta_{\mu\sigma} \Gamma^\sigma_{\lambda\nu} - \Gamma^\delta_{\mu\nu} \Gamma^\sigma_{\lambda\sigma} + \Gamma^\delta_{\mu\sigma\nu} - \Gamma^\delta_{\nu\mu\sigma},
\]
(3)

\[
R^\delta_{\nu\sigma} = R^\delta_{\sigma\nu} = \Gamma^\delta_{\nu\epsilon} \Gamma^\epsilon_{\delta\sigma} - \Gamma^\delta_{\nu\sigma} \Gamma^\epsilon_{\delta\epsilon} - \Gamma^\delta_{\sigma\nu} \Gamma^\epsilon_{\delta\epsilon},
\]
(4)

Symmetric and antisymmetric Ricci tensors are two sides of the curvature tensor; these two sides give us the opportunity to have symmetric and antisymmetric gravitational field equations.

4. General theory of Relativity

General theory of relativity is the modern theory of gravity; General theory of relativity relate gravitational field to the curvature of space time. Stress energy tensor \( T_{\mu\nu} \) describing the constituents of the universe; it is the source of gravitational field in general theory of relativity. In the presence of permanent gravitational field the symmetric gravitational field equation is
\[
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu},
\]
(5)
The Ricci scalar $R = g^{\mu\nu} R_{\mu\nu}$ is the contraction of Ricci tensor; $G$ is the gravitational constant. Einstein-Hilbert action for gravity given by
\[ S = \int \mathcal{L}_{GR} d^4V = \int -\frac{c^4}{16\pi G} (R - 2\Lambda)\sqrt{-g} \, d^4x; \]
where $dV = \sqrt{-g} \, d^4x$ is invariant volume element.

Gravity Lagrangian defined by
\[ \mathcal{L}_{GR} = -\frac{c^4}{16\pi G} (R - 2\Lambda). \tag{6} \]
Gravity Lagrangian is a combination of Ricci scalar and cosmological constant.

5. Electrodynamics

Electrodynamics Lagrangian is given by
\[ \mathcal{L}_{QED} = -\frac{1}{4} F^{\nu\sigma} F_{\nu\sigma} + \bar{\psi}\left( i \gamma^\nu D^\nu - m \right)\psi, \tag{7} \]
where $F^{\nu\sigma} = \partial^\nu A^\sigma - \partial^\sigma A^\nu$ is the electromagnetic field strength tensor, $D^\nu = \partial^\nu + ieA^\nu$ is the gauge contravariant derivative, $e$ is coupling constant, $\psi$ matter field quark and lepton in standard model, $\bar{\psi} = \gamma_0\psi^\dagger$ is their adjoint and $\gamma^\nu$ is the four Dirac matrices with $(\nu=0,1 \ldots 3)$. The electromagnetic field strength tensor ($F^{\nu\sigma}$) is given by
\[ F^{\nu\sigma} = \begin{bmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{bmatrix} \quad \text{and} \quad F_{\nu\sigma} = \begin{bmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{bmatrix} \]
their lowered index counterpart.

Electrodynamics Lagrangian contains three parts, the first part is the electromagnetic field Lagrangian and given by
\[ \mathcal{L}^{\text{em}} = -\frac{1}{4} F^{\nu\sigma} F_{\nu\sigma}. \tag{8} \]
Canonical energy momentum tensor for electromagnetic field Lagrangian is
\[ \theta^{\text{em}}_{\nu\sigma} = -\frac{\partial \mathcal{L}^{\text{em}}}{\partial (\partial^\nu A^\mu)} \partial^\sigma A^\mu - g_{\nu\sigma} \mathcal{L}^{\text{em}}. \tag{9} \]
Using the identity $\frac{\partial (F^{\nu\sigma} F_{\nu\sigma})}{\partial (\partial^\nu A^\mu)} = 4 F_{\nu\sigma}$, we find
\[ \theta^{\text{em}}_{\nu\sigma} = -F_{\nu\mu} F^\mu_{\sigma} + \frac{1}{4} g_{\nu\sigma} F^{\mu\delta} F_{\mu\delta}. \tag{10} \]
Equation (10) is not antisymmetric and also is not gauge invariant due to the asymmetric Tensor \((-F_{\nu\sigma}F_{\sigma}^\nu)\) [16]. For this, suppose that it is divided into two parts symmetric part appears in symmetric equations and antisymmetric part appears in antisymmetric equations.

\[-F_{\nu\mu}F_{\sigma}^\mu = \partial^\sigma \chi_{\sigma\nu\mu} - F_{\nu\mu}F_{\sigma}^\mu - \partial^\sigma \chi_{\sigma\nu\mu}.\]  \hspace{1cm} (11)

The divergence tensor is arbitrary antisymmetric tensor in their first two indices \((\Theta_{\sigma\mu\nu})\); it is constructed from electromagnetic field strength tensor \((F_{\nu\sigma})\) and electromagnetic vector potential \((A_\mu)\); equation (11) in terms of this definition will be

\[-F_{\nu\mu}F_{\sigma}^\mu = \partial^\sigma (F_{\nu\sigma}A_\mu) - F_{\nu\mu}F_{\sigma}^\mu - \partial^\sigma (F_{\nu\sigma}A_\mu).\]  \hspace{1cm} (12)

Employing the Maxwell equation we obtain

\[-F_{\nu\mu}F_{\sigma}^\mu = -j_\nu A_\mu - F_{\nu\mu}F_{\sigma}^\mu + j_\nu A_\mu.\]  \hspace{1cm} (13)

The antisymmetric stress energy tensor for electromagnetic field can be written in the form

\[T_{\nu\sigma}^{e.m} = j_\nu A_\mu + \frac{1}{4} g_{\nu\sigma} F_{\delta\lambda} F^{\delta\lambda}.\]  \hspace{1cm} (14)

If we multiplied this equation by \(\left(\frac{8\pi G}{c^4}\right)\), we find

\[
\frac{8\pi G}{c^4} T_{\nu\sigma}^{e.m} = \frac{8\pi G}{c^4} j_\nu A_\mu + \frac{8\pi G}{c^4} g_{\nu\sigma} \left[\frac{1}{4} F_{\delta\lambda} F^{\delta\lambda}\right].
\]  \hspace{1cm} (15)

The second part in electrodynamics Lagrangian is Dirac Lagrangian and given by

\[L^{\text{Dirac}} = \bar{\psi}(i \gamma_\nu \partial^\nu - m)\psi,\]  \hspace{1cm} (16)

The canonical energy momentum tensor for Dirac Lagrangian defined by

\[
\theta_{\nu\sigma}^{\text{Dirac}} = \frac{\partial L^{\text{Dirac}}}{\partial (\partial^\nu \psi)} \partial_\sigma \psi + \frac{\partial L^{\text{Dirac}}}{\partial (\partial^\nu \psi^\dagger)} \partial_\sigma \psi^\dagger - g_{\nu\sigma} L^{\text{Dirac}}.
\]  \hspace{1cm} (17)

\[
\theta_{\nu\sigma}^{\text{Dirac}} = \bar{\psi} i \gamma_\nu \partial_\sigma \psi - g_{\nu\sigma} \bar{\psi} (i \gamma_\lambda \partial^\lambda - m)\psi.
\]  \hspace{1cm} (18)

The canonical energy momentum tensor that has been presented in this equation has some disagreeable properties it is not antisymmetric and also is not gauge invariant due to the symmetric part \((\bar{\psi} i \gamma_\nu \partial_\sigma \psi)\). For this, the antisymmetric stress energy tensor can be written as the canonical energy momentum tensor minus this symmetric part;

\[T_{\nu\sigma}^{\text{Dirac}} = \theta_{\nu\sigma}^{\text{Dirac}} - \bar{\psi} i \gamma_\nu \partial_\sigma \psi,\]  \hspace{1cm} (19)

\[T_{\nu\sigma}^{\text{Dirac}} = -g_{\nu\sigma} \bar{\psi} (i \gamma_\lambda \partial^\lambda - m)\psi.\]  \hspace{1cm} (20)
Multiplying equation (20) by \( \left( \frac{8\pi G}{c^4} \right) \), we have
\[
\frac{8\pi G}{c^4} T_{\nu\sigma}^{\text{Dirac}} = -\frac{8\pi G}{c^4} g_{\nu\sigma} \bar{\psi}(i \gamma_\lambda \partial^\lambda - m)\psi .
\]  
(21)

Third part in electrodynamics Lagrangian is the interaction’s part and given by
\[
\mathcal{L}^{\text{int}} = -e \bar{\psi} \gamma^\mu A^\mu ,
\]  
(22)

The canonical energy momentum tensor for interaction’s part given by
\[
\theta^{\text{int}}_{\nu\sigma} = g_{\nu\sigma} \bar{\psi} \gamma^\lambda A^\lambda ;
\]  
(23)

Antisymmetry stress energy tensor for interaction’s part is
\[
T^{\text{int}}_{\nu\sigma} = g_{\nu\sigma} \bar{\psi} \gamma^\lambda A^\lambda .
\]  
(24)

Antisymmetric Stress energy tensor for interaction’s part is the same canonical energy momentum tensor; multiplying the previous equation by \( \left( \frac{8\pi G}{c^4} \right) \), we find
\[
\frac{8\pi G}{c^4} T^{\text{int}}_{\nu\sigma} = \frac{8\pi G}{c^4} g_{\nu\sigma} \bar{\psi} \gamma^\lambda A^\lambda .
\]  
(25)

If we added eq (15), (21) to eq (25), we have
\[
\frac{8\pi G}{c^4} \left[ T_{\nu\sigma}^{\text{int}} + T_{\nu\sigma}^{\text{int}} + T_{\nu\sigma}^{\text{Dirac}} \right] = \frac{8\pi G}{c^4} j_\nu A_\mu + \frac{8\pi G}{c^4} g_{\nu\sigma} \left[ \frac{1}{4} F_{\delta\lambda} F^{\delta\lambda} + e \bar{\psi} \gamma^\lambda A^\lambda - \bar{\psi}(i \gamma_\lambda \partial^\lambda - m)\psi \right] .
\]  
(26)

If gauge contravariant derivative definition used in the previous equation, we find
\[
\frac{8\pi G}{c^4} \left[ T_{\nu\sigma}^{\text{ext}} + T_{\nu\sigma}^{\text{int}} + T_{\nu\sigma}^{\text{Dirac}} \right] = \frac{8\pi G}{c^4} j_\nu A_\mu + \frac{2\pi G}{c^4} F_{\delta\lambda} F^{\delta\lambda} - \frac{8\pi G}{c^4} g_{\nu\sigma} \left[ \bar{\psi}(i \gamma_\lambda D^\lambda - m)\psi \right] ,
\]  
(27)

\[
\frac{8\pi G}{c^4} \left[ T_{\nu\sigma}^{\text{ext}} + T_{\nu\sigma}^{\text{int}} + T_{\nu\sigma}^{\text{Dirac}} \right] = \frac{8\pi G}{c^4} j_\nu A_\mu + \frac{2\pi G}{c^4} F_{\delta\lambda} F^{\delta\lambda} - \frac{1}{2} g_{\nu\sigma} \left[ 16\pi G \frac{1}{c^4} \bar{\psi}(i \gamma_\lambda D^\lambda - m)\psi \right] ,
\]  
(28)

\[
\frac{8\pi G}{c^4} T_{\nu\sigma} = R_{\nu\sigma} + \Lambda g_{\nu\sigma} - \frac{1}{2} R g_{\nu\sigma} .
\]  
(29)

Antisymmetric gravitational field equation is gauge theory as well as Einstein’s gravitational field equation; antisymmetric stress energy tensor can be written as
\[
T_{\nu\sigma} = \left[ T_{\nu\sigma}^{\text{Dirac}} + T_{\nu\sigma}^{\text{int}} + T_{\nu\sigma}^{\text{ext}} \right] ,
\]  
(30)

Ricci scalar can be written in the form
\[ R = \frac{16\pi G}{c^4} \mathcal{L}_{\text{QED}} \]  
\[ \text{(31)} \]

Cosmological constant is
\[ \Lambda = \frac{2\pi G}{c^4} F_{\delta\lambda} F_{\delta\lambda}, \]  
\[ \text{(32)} \]

And antisymmetric Ricci tensor is
\[ R_{\nu\sigma} = \frac{8\pi G}{c^4} j_\nu A_{\mu}. \]  
\[ \text{(33)} \]

Substituting by equation (31), (32) into eq (6), we have
\[ \mathcal{L}_{GR} = \frac{c^4}{16\pi G} \left[ \frac{16\pi G}{c^4} \mathcal{L}_{\text{QED}} \right] - \frac{c^4}{8\pi G} \left[ \frac{2\pi G}{c^4} F_{\delta\lambda} F_{\delta\lambda} \right] - \varphi(i \gamma_\lambda D^\lambda - m) \varphi = \frac{1}{4} F_{\delta\lambda} F_{\delta\lambda} = \mathcal{L}_{QED}. \]  
\[ \text{(34)} \]

Gravity Lagrangian is the same electrodynamics Lagrangian; gravity and electromagnetism are two sides of the same Lagrangian.

If we multiplied equation (13) by \( \left( \frac{8\pi G}{c^4} \right) \), we find
\[ \frac{8\pi G}{c^4} F_{\nu\mu} F_{\sigma}^{\mu} = R_{\mu\nu} + R_{\nu\sigma}; \]  
\[ \text{(35)} \]

The symmetric Ricci tensor is given by
\[ R_{\mu\nu} = \frac{8\pi G}{c^2} \left[ -j_\nu A_{\mu} - F_{\nu\sigma} F_{\sigma}^{\mu} \right]. \]  
\[ \text{(36)} \]

The equations (33), (36) are new definitions for symmetric and antisymmetric Ricci tensors. If we substituted by equation (36) into equation (3), we find
\[ \frac{8\pi G}{c^2} F_{\nu\mu} F_{\sigma}^{\mu} - \frac{8\pi G}{c^2} j_\nu A_{\mu} = \Gamma_{\delta\mu}^{\lambda} \Gamma_{\nu}^{\delta} - \Gamma_{\mu\nu}^{\lambda} \Gamma_{\delta\nu}^{\delta} + \Gamma_{\mu\nu,\delta}^{\delta} - \Gamma_{\mu\nu,\delta}, \]  
\[ \text{(37)} \]

Substituting by equation (33) into equation (4), we find
\[ \frac{8\pi G}{c^4} j_\nu A_{\mu} = \Gamma_{\delta\sigma,\nu} - \Gamma_{\delta\nu,\sigma}, \]  
\[ \text{(38)} \]

This also can be written as
\[ \frac{8\pi G}{c^4} j_\nu A_{\mu} = \partial_\nu \partial_\sigma \log \sqrt{-g} - \partial_\sigma \partial_\nu \sqrt{-g}. \]  
\[ \text{(39)} \]

Equation (37) can be divided into two equations as follow
\[-\frac{8\pi G}{c^2} F_{\nu\mu} F_{\sigma}^{\mu} = \Gamma^\lambda_{\mu\delta} \Gamma^\delta_{\lambda\nu} - \Gamma^\lambda_{\nu\mu} \Gamma^\delta_{\lambda\delta}, \quad (40)\]
\[-\frac{8\pi G}{c^2} j_\nu A_\mu = \Gamma^\delta_{\mu\delta,\nu} - \Gamma^\delta_{\mu\nu,\delta}. \quad (41)\]

Equation (41) can be written as
\[-\frac{8\pi G}{c^2} j_\nu A_\mu = \partial_\nu \partial_\mu \log \sqrt{-g} - \partial_\nu \Gamma^\delta_{\mu\nu} , \quad (42)\]
\[\partial_\nu \partial_\mu \log \sqrt{-g} + \frac{8\pi G}{c^2} j_\nu A_\mu = \partial_\nu \Gamma^\delta_{\mu\nu} , \quad (43)\]
\[\Gamma^\delta_{\mu\nu} = \frac{1}{2} g^{\delta\sigma} \left( \partial_\nu g_{\sigma\mu} + \partial_\mu g_{\sigma\nu} - \partial_\sigma g_{\mu\nu} \right) = \frac{1}{2} g^{\delta\sigma} \left( \partial_\sigma g_{\nu\mu} - \partial_\nu g_{\mu\sigma} \right) , \quad (44)\]
\[g_{\mu\sigma} = g_{\sigma\mu} = g^{\mu\sigma} = g^{\nu\mu} g^{\nu\sigma} = 0 , \quad (45)\]
\[\partial_\nu \Gamma^\delta_{\mu\nu} = \frac{1}{2} \partial_\nu g^{\delta\sigma} \partial_\mu g_{\sigma\nu} - \frac{1}{2} \partial_\nu g^{\delta\sigma} \partial_\sigma g_{\mu\nu} = \frac{1}{2} \partial_\nu \partial_\mu g^{\delta\sigma} g_{\sigma\nu} - \frac{1}{2} \partial_\nu \partial_\sigma g^{\delta\sigma} g_{\mu\nu} . \quad (46)\]

Substitute by eq (46) into eq (43), we find
\[\partial_\nu \partial_\mu \log \sqrt{-g} + \frac{8\pi G}{c^2} j_\nu A_\mu = \frac{1}{2} \partial_\nu \partial_\mu g^{\delta\sigma} - \frac{1}{2} \partial_\nu \partial_\sigma g^{\delta\sigma} g_{\mu\nu} . \quad (47)\]

Equating the first part on L.H.S by the first part on R.H.S, we find
\[\partial_\nu \partial_\mu \log \sqrt{-g} = \frac{1}{2} \partial_\nu \partial_\mu g^{\delta\sigma} , \quad (48)\]
\[\partial_\nu \log \sqrt{-g} = \frac{1}{2} \partial_\nu g^{\delta\sigma} . \quad (49)\]

In eq (47) if we equate the second part on L.H.S by the second part on R.H.S, we find
\[\frac{8\pi G}{c^2} j_\nu A_\mu = -\frac{1}{2} \partial_\nu \partial_\sigma g^{\delta\sigma} g_{\mu\nu} . \quad (50)\]

Equating (50) with (39), we find
\[\partial_\nu \partial_\sigma \log \sqrt{-g} - \partial_\nu \partial_\sigma \sqrt{-g} = -\frac{1}{2} \partial_\nu \partial_\sigma g^{\delta\sigma} g_{\mu\nu} . \quad (51)\]

Gravitational field tensor in terms of Christoffel’s symbols can be written as
\[- \frac{8\pi G}{c^4} F_{\mu\nu} F_{\sigma}^{\mu} = \Gamma^\lambda_{\mu\delta} \Gamma^\sigma_{\lambda\nu} - \Gamma^\delta_{\mu\nu} \partial_\lambda \log \sqrt{-g}, \]  

(52)

\[ \Gamma^\lambda_{\mu\delta} = \frac{1}{2} g^{\lambda \sigma} \left( \partial_\sigma g_{\mu\delta} + \partial_\mu g_{\sigma\delta} - \partial_\delta g_{\mu\sigma} \right) = \frac{1}{2} g^{\lambda \sigma} \left( \partial_\mu g_{\sigma\delta} - \partial_\sigma g_{\mu\delta} \right), \]  

(53)

\[ \Gamma^\delta_{\lambda\nu} = \frac{1}{2} g^{\delta \sigma} \left( \partial_\sigma g_{\lambda\nu} + \partial_\lambda g_{\sigma\nu} - \partial_\nu g_{\lambda\sigma} \right), \]  

(54)

\[ \Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda \sigma} \left( \partial_\sigma g_{\mu\nu} + \partial_\mu g_{\sigma\nu} - \partial_\nu g_{\mu\sigma} \right) = -\frac{1}{2} g^{\lambda \sigma} \partial_\mu g_{\nu\sigma} - \frac{1}{2} g^{\lambda \sigma} \partial_\sigma g_{\mu\nu}, \]  

(55)

\[ \Gamma^\lambda_{\mu\delta} \Gamma^\delta_{\lambda\nu} = \left[ \frac{1}{2} g^{\lambda \sigma} \left( \partial_\mu g_{\sigma\delta} - \partial_\delta g_{\mu\sigma} \right) \right] \left[ \frac{1}{2} g^{\delta \sigma} \left( \partial_\nu g_{\sigma\lambda} + \partial_\lambda g_{\sigma\nu} - \partial_\sigma g_{\lambda\nu} \right) \right] \]

\[ = \frac{1}{4} \left[ g^{\lambda \sigma} \partial_\mu g_{\sigma\delta} - g^{\lambda \sigma} \partial_\delta g_{\mu\sigma} \right] \left[ g^{\delta \sigma} \partial_\nu g_{\sigma\lambda} + g^{\delta \sigma} \partial_\lambda g_{\sigma\nu} - g^{\delta \sigma} \partial_\sigma g_{\lambda\nu} \right] \]

\[ = \frac{1}{4} \left[ g^{\lambda \sigma} \partial_\mu g_{\sigma\delta} g^{\delta \sigma} \partial_\nu g_{\sigma\lambda} + g^{\lambda \sigma} \partial_\mu g_{\sigma\delta} g^{\delta \sigma} \partial_\lambda g_{\sigma\nu} - g^{\lambda \sigma} \partial_\sigma g_{\mu\delta} g^{\delta \sigma} \partial_\nu g_{\sigma\lambda} - g^{\lambda \sigma} \partial_\sigma g_{\mu\delta} g^{\delta \sigma} \partial_\lambda g_{\sigma\nu} + g^{\lambda \sigma} \partial_\sigma g_{\mu\delta} g^{\delta \sigma} \partial_\sigma g_{\lambda\nu} \right] \]

\[ = \frac{1}{4} \left[ g^{\lambda \sigma} \partial_\mu g_{\sigma\delta} g^{\delta \sigma} \partial_\nu g_{\sigma\lambda} + g^{\lambda \sigma} \partial_\mu g_{\sigma\delta} g^{\delta \sigma} \partial_\lambda g_{\sigma\nu} - g^{\lambda \sigma} \partial_\sigma g_{\mu\delta} g^{\delta \sigma} \partial_\nu g_{\sigma\lambda} - g^{\lambda \sigma} \partial_\sigma g_{\mu\delta} g^{\delta \sigma} \partial_\lambda g_{\sigma\nu} \right] \]

\[ = \frac{1}{4} \left[ \partial_\mu g^{\lambda \sigma} \partial_\nu g_{\sigma\lambda} + \partial_\mu g^{\lambda \sigma} \partial_\lambda g_{\sigma\nu} - \partial_\mu g^{\lambda \sigma} \partial_\sigma g_{\lambda\nu} \right] \]

\[ = \frac{1}{4} \left[ \partial_\mu g^{\lambda \sigma} \partial_\nu g_{\sigma\lambda} + \partial_\mu g^{\lambda \sigma} \partial_\lambda g_{\sigma\nu} - \partial_\mu g^{\lambda \sigma} \partial_\sigma g_{\lambda\nu} \right] \]

\[ = \frac{1}{4} \left[ \partial_\mu g^{\lambda \sigma} \partial_\nu g_{\sigma\lambda} + \partial_\mu g^{\lambda \sigma} \partial_\lambda g_{\sigma\nu} - \partial_\mu g^{\lambda \sigma} \partial_\sigma g_{\lambda\nu} \right] \]

\[ = \frac{1}{4} \left[ \partial_\mu g^{\lambda \sigma} \partial_\nu g_{\sigma\lambda} + \partial_\mu g^{\lambda \sigma} \partial_\lambda g_{\sigma\nu} - \partial_\mu g^{\lambda \sigma} \partial_\sigma g_{\lambda\nu} \right] \]

\[ = \frac{1}{4} \left[ \partial_\mu g^{\lambda \sigma} \partial_\nu g_{\sigma\lambda} + \partial_\mu g^{\lambda \sigma} \partial_\lambda g_{\sigma\nu} - \partial_\mu g^{\lambda \sigma} \partial_\sigma g_{\lambda\nu} \right] \]

Using equation (49), we find
\[ \Gamma^\lambda_{\mu\nu} \Gamma^\mu_{\lambda\nu} = \frac{1}{4} \left[ \partial_\mu \partial_\nu g^A_{\lambda} + 2 \partial_\mu \partial_\nu \log \sqrt{-g} - 2 \partial_\mu \partial_\nu \log \sqrt{-g} \right] \]
\[ = \frac{1}{4} \partial_\mu \partial_\nu g^A_{\lambda} \]  

(57)

Equation (52) can be written as
\[ -8\pi G c^2 F_{\nu\mu} F^\sigma_{\mu} = \frac{1}{4} \partial_\mu \partial_\nu g^A_{\lambda} + \frac{1}{2} g^{\lambda\sigma} \partial_\mu \partial_\nu \log \sqrt{-g} g_{\nu\sigma} + \frac{1}{2} g^{\lambda\sigma} \partial_\sigma \partial_\lambda \log \sqrt{-g} g_{\mu\nu} \]  

(58)

And now, let’s construct the antisymmetric metric tensor; electric and magnetic fields are only properties of space time. Magnetic field in vacuum is given by
\[ B = B_{01} e^{i(k_1 x_1 - \omega t)} + B_{02} e^{i(k_2 x_2 - \omega t)} + B_{03} e^{i(k_3 x_3 - \omega t)} \],

(59)

\( \omega = \omega_1 + \omega_2 + \omega_3 \), \( \vec{k} = (k_1, k_2, k_3) \) are the wave frequency and wave vector. In general orthogonal curvilinear coordinates a vector \( \vec{A} \) defined as follow;
\[ \vec{A} = e_i h_i + e_2 h_2 + e_3 h_3 \].

(60)

Let’s suppose that \((B_{01}, B_{02}, B_{03})\) is the unit vector then equate (59) with eq (60), we find 
\[ h_i = e^{i(k_i x_i - \omega t)} \], \( h_2 = e^{i(k_2 x_2 - \omega t)} \) and \( h_3 = e^{i(k_3 x_3 - \omega t)} \); these three coefficients are a curvature of space time in general coordinates.

Using sign as in electromagnetic field strength tensor \( F_{\nu\sigma} \) we see that antisymmetric metric tensor \( (g_{\nu\sigma}) \) in general coordinates as

\[
g_{\nu\sigma} = \begin{bmatrix}
0 & h_1 & h_2 & h_3 \\
-h_1 & 0 & -h_2 & h_3 \\
-h_2 & h_3 & 0 & -h_1 \\
-h_3 & -h_2 & h_1 & 0
\end{bmatrix} = \begin{bmatrix}
0 & e^{i(k_1 x_1 - \omega t)} & e^{i(k_2 x_2 - \omega t)} & e^{i(k_3 x_3 - \omega t)} \\
-e^{i(k_1 x_1 - \omega t)} & 0 & -e^{i(k_2 x_2 - \omega t)} & e^{i(k_3 x_3 - \omega t)} \\
-e^{i(k_2 x_2 - \omega t)} & e^{i(k_1 x_1 - \omega t)} & 0 & -e^{i(k_3 x_3 - \omega t)} \\
-e^{i(k_3 x_3 - \omega t)} & -e^{i(k_2 x_2 - \omega t)} & e^{i(k_1 x_1 - \omega t)} & 0
\end{bmatrix}.
\]

(61)

And now, we will return to the cosmological constant; the cosmological constant can be divided into two parts where
\[ F_{\nu\lambda} F^{\nu\lambda} = -2E^2 + 2B^2; \]
\[ \Lambda = -\frac{4\pi G}{c^4} E^2 + \frac{4\pi G}{c^4} B^2. \]

(62)

The first part can be written as
\[ \Lambda_1 = \frac{8\pi G}{c^4} \rho_i. \]

(63)
First part of the cosmological constant is proportional to density of vacuum electric energy; Density of vacuum electric energy is equivalent to binding energy per nucleon $\frac{B.E.}{A}$; $\frac{B.E.}{A} = -\frac{\Delta m}{A} \cdot c^2$, where $\Delta m = Zm_p + (A-Z)m_n - M_N$, $A$ is atomic mass number, $Z$ is atomic number, $M_N$ is a nucleus mass, $m_p$ is proton mass and $m_n$ is neutron mass [18].

$$\rho_1 = -\frac{1}{2} E^2 = \frac{B.E.}{A}.$$  \hspace{1cm} (64)

First part of cosmological constant is a function of atomic mass number ($A$) and continuous quantity, and it is representing by a curve. The second part can be written as

$$\Lambda_2 = \frac{8 \pi G}{c^4} \rho_2,$$  \hspace{1cm} (65)

$$\rho_2 = \frac{1}{2} B^2 = \frac{B.E.}{A}.$$  \hspace{1cm} (66)

Second part is proportional to density of vacuum magnetic energy. Density of vacuum magnetic energy equal to the absolute value of binding energy per nucleon; second part of cosmological constant is representing by a curve and it is the image of the first part by reflection on the $A$-axis in $AA$-plane. Symmetric gravitational field equation in vacuum is

$$R_{\mu \nu} - \frac{1}{2} R g_{\mu \nu} = -g_{\mu \nu} \Lambda,$$  \hspace{1cm} (67)

$$R_{\mu \nu} = \left( \frac{1}{2} R - \Lambda \right) g_{\mu \nu}.$$  \hspace{1cm} (68)

Substituting by eq (36) into eq (68), we find

$$-\frac{8 \pi G}{c^2} j_{\mu} - \frac{8 \pi G}{c^2} F_{\mu \nu} F_{\sigma \nu} = \left( \frac{1}{2} R - \Lambda \right) g_{\mu \nu}.$$  \hspace{1cm} (69)

Antisymmetric gravitational field equation in vacuum by analogy to symmetric gravitational field equation is

$$R_{\nu \sigma} - \frac{1}{2} R g_{\nu \sigma} = -g_{\nu \sigma} \Lambda,$$  \hspace{1cm} (70)

$$R_{\nu \sigma} = \left( \frac{1}{2} R - \Lambda \right) g_{\nu \sigma}.$$  \hspace{1cm} (71)

Equating eq (71) with eq (33), we have
\[ \frac{8\pi G}{c^4} j_{\nu} A_{\mu} = \left( \frac{1}{2} R - \Lambda \right) g_{\nu\sigma}. \]  

(72)

Equations (69), (72) are two states for vacuum energy and gravitational object transit between them by gaining or losing gravitational radiation; if we added eq (69) into eq (72) we have

\[ \frac{8\pi G}{c^4} F_{\mu\nu} F_{\sigma}^{\mu} = \left( \frac{1}{2} R - \Lambda \right) g_{\nu\sigma} + \left( \frac{1}{2} R - \Lambda \right) g_{\mu\nu}. \]  

(73)

If we compared eq (73) by eq (58), the first part of eq (58) hasn’t comparable one in eq (73) and equal to zero;

\[ \frac{1}{4} \partial_{\mu} \partial_{\nu} G^{\lambda, \lambda} = 0 \]  

(74)

This equation gives a new relation in electromagnetism and differential geometry.

Equating first part of eq (73) by second of (58), we find

\[ \frac{1}{2} g^{\lambda, \sigma} \partial_{\mu} \partial_{\lambda} \log \sqrt{-g} = \frac{1}{2} R - \Lambda \]  

(75)

Equating second part of eq (73) by third part of eq (58), we find

\[ \frac{1}{2} g^{\lambda, \sigma} \partial_{\sigma} \partial_{\lambda} \log \sqrt{-g} = \frac{1}{2} R - \Lambda \]  

(76)

Equating eq (75) by eq (76), we find

\[ \frac{1}{2} g^{\lambda, \sigma} \partial_{\sigma} \partial_{\lambda} \log \sqrt{-g} = \frac{1}{2} g^{\lambda, \sigma} \partial_{\mu} \partial_{\lambda} \log \sqrt{-g}. \]  

(77)

\[ \partial_{\sigma} = \partial_{\mu}. \]  

(78)

This equation gives a relation between two gradient operators in vacuum.

6. Conclusion

We concluded to unification of Gravity and electromagnetism. Differential geometry has been extended by new tensors and operators; metric tensor became four tensors; these tensors are \( g_{\mu\nu}, g_{\nu\sigma}, g_{\sigma\delta}, g_{\delta\lambda} \); the four dimensional gradient operator became six operators; these operators are \( \partial_{\mu}, \partial_{\nu}, \partial_{\sigma}, \partial_{\xi}, \partial_{\delta}, \partial_{\lambda} \). Multiplications of these geometric objects give new relations in differential geometry; this study created new differential geometry analysis undertaken.

7. References


[8] Uwe-Jens Wiese “Classical field theory” Institute for theoretical physics –Bern University-May 25, 2009
[13] Lous Witten “Geometry of gravitation and electromagnetism” Physical Review, volume 115, number 1, July 1, 1959
[14] Matthias Blau “Lecture notes in General Relativity” Albert Einstein center for fundamental physics, institute of theoretical physics, Bern University, CH-3012Bern, Switzerland
[25] David H. Boal “Modern physics from quarks to galaxies”. Physics Department, Simon Fraser University
[33] Sean Carroll “Does the Universe Need God”. Draft to appear in the Blackwell companion to science and Christianity
[36] John A. Macken “The universe is only space time”. Santa rose California; original draft-February 2010; Revision 7.1-May 2013
[38] William O. Staub “Weyl’s theory of the combined gravitational-electromagnetic field”. PhD, Pasadena, California
[41] Leong Ying “Physical null conditions: Diameter of a black hole Singularity”. Doi:10.11648/j.ajmp.s.2015040101.18
[46] Amrit Sorli “Gravity as a result quantum vacuum energy density”. Space life institute, Slovenia; www.spacelife.si
[48] Peter A. Tanner, Michael Chua and Michael F. Reid “Energy transfer by magnetic dipole-magnetic dipole interaction”. Chemical physics letters ,vol 209, No 5,6