Weighted Boundedness of Commutators of Riesz Transforms Associated with Schrödinger Operator*

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Abstract. In this paper, we consider Schrödinger operator $-\Delta + V(x)$ on $\mathbb{R}^n (n \geq 3)$, where $V(x)$ is non-zero, non-negative, and belongs to reverse Hölder classes $B_q$ for some $q \geq \frac{n}{2}$. Let $T_1 = (-\Delta + V)^{-1}V, T_2 = (-\Delta + V)^{-\frac{1}{2}}V^{\frac{1}{2}}, T_3 = (-\Delta + V)^{-\frac{1}{2}}\nabla, T_4 = (-\Delta + V)^{-1}\nabla^2$. We show that the commutators $[b, T_j](j = 1, 2, 3, 4)$ are bounded operators from $L^p(\mu)$ to $L^r(\mu^{1-r})$, when $b \in \text{Lip}_{\beta, \mu}(0 < \beta < 1)$, for some weight function $\mu$. The weighted boundedness of commutators of weighted BMO functions and $T_j(j = 1, 2, 3, 4)$ also are obtained similarly.

Key words: Schrödinger operator; Riesz transforms; Reverse Hölder classes; Weighted Lipschitz function; Commutator

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1 Introduction

Let $P = -\Delta + V(x)$ be the Schrödinger differential operator on $\mathbb{R}^n, n \geq 3$. We assume that $V(x)$ is a non-zero, non-negative potential, and belongs to reverse Hölder classes $B_q$ for some $q \geq \frac{n}{2}$. Let $T_j(j = 1, 2, 3, 4)$ be the Riesz transforms associated to Schrödinger operators, namely $T_1 = (-\Delta + V)^{-1}V, T_2 = (-\Delta + V)^{-\frac{1}{2}}V^{\frac{1}{2}}, T_3 = (-\Delta + V)^{-\frac{1}{2}}\nabla$ and $T_4 = (-\Delta + V)^{-1}\nabla^2$. J. Zhong (see[1]) proved that if $V$ is a non-negative polynomial, $V^2(-\Delta + V)^{-1}, \nabla(-\Delta + V)^{-\frac{1}{2}}$ and $\nabla(-\Delta + V)^{-1}\nabla$ are Calderón-Zygmund operators. Z. Shen generalized these results (see[2]).

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proved that $\nabla (\Delta + V)^{-\frac{1}{2}} \nabla (\Delta + V)^{-\frac{1}{2}} \nabla$ and $\nabla (\Delta + V)^{-1} \nabla$ are Calderón-Zygmund operators, if $V$ belongs to the reverse Hölder class $B_n$, which includes non-negative polynomials and allows some non-smooth potentials. Moreover, Z.Shen also shown $L^p$ boundedness for $T_1, T_2, T_3$ and $T_4$ when $V \in B^+_2$. Recently, Z.Guo, P.Li and L. Peng (see [3]) studied $L^p$ boundedness of commutators $[b, T_j] = bT_j - T_jb (j = 1, 2, 3, 4)$, when $b \in BMO(\mathbb{R}^n)$. Canqing Tang and Bolin Ma (see [4]) proved $L^p$ boundedness of commutators $[b, T_j](j = 1, 2, 3, 4)$, when $b$ is a Lipschitz function. In this paper we will generalize results in [3] and [4] to weighted case.

**Definition 1** A non-negative locally $L^q$ integrable function $V(x)$ on $\mathbb{R}^n$ is said to belong to $B_q(1 < q < \infty)$, if there exists $C > 0$ such that the reverse Hölder inequality

$$(\frac{1}{|B|} \int_B V(y)^q dy)^{\frac{1}{q}} \leq C \left( \frac{1}{|B|} \int_B V(y) dy \right),$$

(1)

holds for every ball $B$ in $\mathbb{R}^n$.

By Hölder inequality, it is easy to see that $B_{q_1} \subset B_{q_2} (q_1 > q_2 > 1)$. One remarkable feature about the $B_q$ class is that, if $V \in B_q$ for some $q > 1$, then there exists $\epsilon > 0$, which depends only on $n$ and the constant $C$ in (1), such that $V \in B_{q+\epsilon}$ (see [5]). It is also well known that, if $V \in B_q(q > 1)$, then $V(x)dx$ is a doubling measure, namely for any $r > 0, x \in \mathbb{R}^n$, there exists a constant $C_0 > 0$ such that

$$\int_{B(y, 2r)} V(y) dy \leq C_0 \int_{B(x, r)} V(y) dy.$$  

(2)

It was proved that if $V \in B_n$, then $T_3$ is a Calderón-Zygmund operator (see [2]). According to M.Paluszyński’s classical result in [6], if $T$ is a Calderón-Zygmund operator, $b \in Lip_\beta (0 < \beta < 1)$ if and only if the commutator $[b, T]$ is bounded from $L^p$ to $L^q$, where $1 < p < q < \infty$, $\frac{1}{q} = \frac{1}{p} - \frac{\beta}{n}$, then the commutator $[b, T_3]$ and $[b, T_4]$ are bounded from $L^p$ to $L^q$, where $1 < p < q < \infty$, $\frac{1}{q} = \frac{1}{p} - \frac{\beta}{n}$. Bei Hu, Jiajun Gu in [7] shown that for $\mu \in A_1$, $b \in Lip_{\beta, \mu}$ if and only if the commutator $[b, T]$ is bounded from $L^p(\mu)$ to $L^q(\mu^{1-q})$, where $1 < p < q < \infty, 0 < \beta < 1$, and $\frac{1}{q} = \frac{1}{p} - \frac{\beta}{n}$. However, in [3], the authors shown these kernels had no smoothness of C-Z kernel due to $V \in B_q$, for some $q > \frac{n}{2}$ and they discovered that the kernels have some other kind of smoothness.

**Definition 2[3]** $K(x, y)$ is said to satisfy $H(m)$ for some $m \geq 1$, if there exists a constant $C \geq 0$ such that, $\forall l > 0, x, x_0 \in \mathbb{R}^n$ with $|x - x_0| \leq l$, then

$$\sum_{k=0}^{\infty} k(2^k l)^\frac{m}{m'} \left( \int_{2^k l \leq |y-x_0| < 2^{k+1} l} |K(x, y) - K(x_0, y)|^m dy \right) < C,$$  

(3)

where $\frac{1}{m} + \frac{1}{m'} = 1$.  

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We use 2k of it, then $K(x, y)$ satisfies $H(m)$, for $m \geq 1$. In [3], the authors proved that these kernels of $T_j (j = 1, 2, 3)$ satisfy $H(m)$, $m$ in different ranges respectively.

We give some notations. A non-negative function $\mu$ defined on $\mathbb{R}^n$ is called weight if it is locally integral. A weighted $\mu$ is said to belong to Muckenhoupt class $A_p(\mathbb{R}^n)$ for $1 < p < \infty$, if there exists a constant $C$ such that

$$\frac{1}{|B|} \int_B \mu(x)dx \left( \frac{1}{|B|} \int_B \mu(x)\frac{1}{r^p} dx \right)^{(p-1)} \leq C$$

holds for every ball $B \subset \mathbb{R}^n$; The class $A_1(\mathbb{R}^n)$ is defined replacing the above inequality by

$$\frac{1}{|B|} \int_B \mu(x)dx \leq C \mu(x), \quad a.e. x \in \mathbb{R}^n$$

for every ball $B(\exists x) \subset \mathbb{R}^n$ (see [8]). It was well known that $A_{p_1} \subset A_{p_2}$, for $1 \leq p_1 \leq p_2 < \infty$ and $\mu \in A_1$ implies $\mu^{1-p} \in A_p (1 < p < \infty)$. We note that $A_\infty = \bigcup_{p \geq 1} A_p$.

**Definition 3** We will say that a locally integrable function $f(x)$ belongs to the weighted $Lip_{\beta, \mu}^p$ for $1 \leq p < \infty, 0 < \beta < 1$ and $\mu \in A_\infty(\mathbb{R}^n)$, that is

$$\sup_{B} \frac{1}{\mu(B)^{\frac{\beta}{p}}} \left[ \frac{1}{\mu(B)} \int_B |f(y) - f_B|^p \mu(y)^{1-p} dy \right]^{\frac{1}{p}} \leq C < \infty,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$.

Modulo constants, the Banach space of such function is denoted by $Lip_{\beta, \mu}^p$. The smallest bound $C$ satisfying conditions above is then taken to the norm of $f$ in these spaces, and is denoted by $\|f\|_{Lip_{\beta, \mu}^p}$. Put $Lip_{\beta, \mu} = Lip_{\beta, \mu}^1$. Obviously, for the case $\mu = 1$, the $Lip_{\beta, \mu}$ is the classical $Lip_{\beta}$ space.

Two basic facts about $Lip_{\beta, \mu}^p$ may be in order (see [9]).

For $\mu \in A_1,
\|f\|_{Lip_{\beta, \mu}^p} \sim \|f\|_{Lip_{\beta, \mu}}$, $\forall p > 1$. (7)

We use $2kB$ to denote the ball with the same center as $B$ but with $2k$ times radius of it, then

$$|f_{2kB} - f_B| \leq C k \mu(B)^{\frac{\beta}{p}} \|f\|_{Lip_{\beta, \mu}}.$$ (8)

We also denote the fractional weighted maximum function of function $f(x)$ by

$$M_{\beta, \mu, r} f(x) = \sup_{x \in B} \left( \frac{1}{\mu(B)^{1 - \frac{\beta}{p}}} \int_B |f(y)|^r \mu(y) dy \right)^{\frac{1}{r}}, \quad 0 < r << \frac{n}{\beta}.$$ (9)

Our main theorems are as follows.

**Theorem 1** Suppose $V \in B_q$ for some $q \geq \frac{n}{2}$, $r = \frac{1}{p} - \frac{\beta}{n}$ for $0 < \beta < 1$ and $q' < p < r < \infty$. Let $\mu \in A_1(\mathbb{R}^n) \cap B_{q_0}$, where $\epsilon_0 = \frac{(s_1 - 1)q'}{s_1 - q'}$ for some $s_1 \in (q', s)$ and
\(s \in (q', \frac{n}{\beta})\). Assume \(\mu^{1-p} \in A_{\frac{n}{q}}\). For every \(b \in \text{Lip}_{\beta, \mu}\), there exists a constant \(C_p\) such that

\[
\| [b, T_1] \|_{L^r(\mu^{1-r})} \leq C_p \| b \|_{\text{Lip}_{\beta, \mu}} \| f \|_{L^p(\mu)}.
\]

**Theorem 2** Suppose \(V \in B_q\) for some \(q \geq \frac{n}{2}\), \(\frac{1}{r} = \frac{1}{p} - \frac{\beta}{n}\) for \(0 < \beta < 1\) and \((2q)' < p < r < \infty\). Let \(\mu \in A_1(\mathbb{R}^n) \cap B_{\epsilon_0}\), where \(\epsilon_0 = \frac{(s_1-1)(2q)'}{s_1-2(2q)'}\) for some \(s_1 \in ((2q)', s)\) and \(s \in ((2q)', \frac{n}{\beta})\). Assume \(\mu^{1-p} \in A_{\frac{n}{\beta}}\). For every \(b \in \text{Lip}_{\beta, \mu}\), there exists a constant \(C_p\) such that

\[
\| [b, T_2] \|_{L^r(\mu^{1-r})} \leq C_p \| b \|_{\text{Lip}_{\beta, \mu}} \| f \|_{L^p(\mu)}.
\]

**Theorem 3** Suppose \(V \in B_q\) for some \(q \geq \frac{n}{2}\), \(\frac{1}{r} = \frac{1}{p} - \frac{\beta}{n}\) for \(0 < \beta < 1\), \(\frac{1}{q} = \frac{1}{n}\) and \(p_0' < p < r < \infty\). Let \(\mu \in A_1(\mathbb{R}^n) \cap B_{\epsilon_0}\), where \(\epsilon_0 = \frac{(s_1-1)p_0' s_1}{s_1-p_0'}\) for some \(s_1 \in (p_0', s)\) and \(s \in (p_0', \frac{n}{\beta})\). Assume \(\mu^{1-p} \in A_{\frac{n}{p_0}}\). For every \(b \in \text{Lip}_{\beta, \mu}\), then

\[
\| [b, T_3] \|_{L^r(\mu^{1-r})} \leq C_p \| b \|_{\text{Lip}_{\beta, \mu}} \| f \|_{L^p(\mu)}.
\]

Let \(T^*\) denote adjoint operator of \(T\). We know that \(T_1^* = V(-\Delta + V)^{-\frac{1}{2}}, T_2^* = V^\frac{1}{2}(-\Delta + V)^{-\frac{1}{2}}, T_3^* = -\nabla(-\Delta + V)^{-\frac{1}{2}}\). By duality, under the same assumptions as in Theorem 1, 2 and 3, we can easily obtain that

\[
\| [b, T_1^*] \|_{L^r(\mu^{1-r})} \leq C_p \| b \|_{\text{Lip}_{\beta, \mu}} \| f \|_{L^p(\mu)}, \quad 1 < p < q,
\]

\[
\| [b, T_2^*] \|_{L^r(\mu^{1-r})} \leq C_p \| b \|_{\text{Lip}_{\beta, \mu}} \| f \|_{L^p(\mu)}, \quad 1 < p < 2q,
\]

\[
\| [b, T_3^*] \|_{L^r(\mu^{1-r})} \leq C_p \| b \|_{\text{Lip}_{\beta, \mu}} \| f \|_{L^p(\mu)}, \quad 1 < p < p_0,
\]

where \(\frac{1}{p_0} = \frac{1}{q} - \frac{1}{n}\).

We know that \(T_4^* = \nabla^2(-\Delta + V)^{-\frac{1}{2}}\). By Theorem 1, we conclude that

**Corollary 1** Suppose \(V \in B_q\) for some \(q \geq \frac{n}{2}\), \(\frac{1}{r} = \frac{1}{p} - \frac{\beta}{n}\), \(0 < \beta < 1\) and \(1 < p < r < \infty\). Let \(\mu \in A_1(\mathbb{R}^n) \cap B_{\epsilon_0}\), where \(\epsilon_0 = \frac{(s_1-1)q'}{s_1-q}\) for some \(s_1 \in (q', s)\) and \(s \in (q', \frac{n}{\beta})\). Assume \(\mu^{1-p} \in A_{\frac{n}{q}}\). For every \(b \in \text{Lip}_{\beta, \mu}\), we have

\[
q' < p < \infty, \quad \| [b, T_4^*] f \|_{L^r(\mu^{1-r})} \leq C_p \| b \|_{\text{Lip}_{\beta, \mu}} \| f \|_{L^p(\mu)}
\]

and

\[
1 < p < q, \quad \| [b, T_4^*] f \|_{L^r(\mu^{1-r})} \leq C_p \| b \|_{\text{Lip}_{\beta, \mu}} \| f \|_{L^p(\mu)}.
\]

**Remark 1** The results in [4] are the special cases of Theorem 1, 2 and Corollary 1, when \(\mu \equiv 1\).

We also consider the commutators of Riesz Transforms Associated to Schrödinger Operator and weighted BMO function.
\textbf{Definition 4} Let $1 \leq p < \infty$, and $\mu \in A_\infty(\mathbb{R}^n)$. We say a local integrable function $f(x)$ belongs to $BMO_\mu^p$, if there exists a constant $C > 0$ such that
\[
\sup_B \left[ \frac{1}{\mu(B)} \int_B |f(y) - f_B|^p \mu(y)^{1-p} dy \right]^{\frac{1}{p}} \leq C < \infty,
\]
where the supremum is taken over all balls $B \subset \mathbb{R}^n$.

Modulo constants, the Banach space of such function is denoted by $BMO_\mu^p$. The smallest bound $C$ satisfying conditions above is then taken to the norm of $f$ in these spaces, and is denoted by $\|f\|_{BMO_\mu^p}$. Put $BMO_\mu = BMO_\mu^1$. Obviously, for the case $\mu = 1$, the $BMO_\mu$ is the classical $BMO$ space.

Two basic facts about $BMO_\mu$ may be in order (see [9]).
\[
\mu \in A_1, \quad \|f\|_{BMO_\mu^p} \sim \|f\|_{BMO_\mu}, \quad \forall p > 1. \tag{10}
\]
We use $2^k B$ to denote the ball with the same center as $B$ but with $2^k$ times radius of it, then
\[
|f_{2^k B} - f_B| \leq C(k + 1)\|f\|_{BMO_\mu}. \tag{11}
\]
Applying the same proving routine of theorems and corollaries above, we can easily obtain the following theorems and corollary.

\textbf{Theorem 4} Suppose $V \subset B_q$ for some $q \geq \frac{n}{2}$, $q' \leq p < \infty$. Let $\mu \in A_1(\mathbb{R}^n) \cap B_{e_0}$, where $e_0 = \frac{(s_1-1)q'}{s_1-q'}$ with some $s_1 \in (q', s)$ and $s > q'$. Assume $\mu^{1-p} \in A_{\frac{p}{p'}}$. For every $b \in BMO_\mu$, there exists a constant $C_{p'}$ such that
\[
\|[b, T_1]f\|_{L^p(\mu^{1-p})} \leq C_{p'}\|b\|_{BMO_\mu}\|f\|_{L^p(\mu)}. \tag{12}
\]

\textbf{Theorem 5} Suppose $V \subset B_q$ for some $q \geq \frac{n}{2}$, $(2q)' < p < \infty$. Let $\mu \in A_1(\mathbb{R}^n) \cap B_{e_0}$, where $e_0 = \frac{(s_1-1)(2q)}{s_1-(2q)}$ with some $s_1 \in ((2q)', s)$ and $s > (2q)'$. Assume $\mu^{1-p} \in A_{\frac{p}{p'}}$. For every $b \in BMO_\mu$, there exists a constant $C_{p'}$ such that
\[
\|[b, T_2]f\|_{L^p(\mu^{1-p})} \leq C_{p'}\|b\|_{BMO_\mu}\|f\|_{L^p(\mu)}. \tag{13}
\]

\textbf{Theorem 6} Suppose $V \subset B_q$ for some $q \geq \frac{n}{2}$, $\frac{1}{p} = \frac{1}{q} - \frac{1}{n}$ and $p_0' < p < r < \infty$. Let $\mu \in A_1(\mathbb{R}^n) \cap B_{e_0}$, where $e_0 = \frac{(s_1-1)p_0'}{s_1-p_0}$ with some $s_1 \in (p_0', s)$ and $s > p_0'$. Assume $\mu^{1-p} \in A_{\frac{p}{p_0'}}$. For every $b \in BMO_\mu$, there exists a constant $C_p$ such that
\[
\|[b, T_3]f\|_{L^r(\mu^{1-p})} \leq C_p\|b\|_{BMO_\mu}\|f\|_{L^r(\mu)}. \tag{14}
\]

\textbf{Corollary 3} Suppose $V \subset B_q$ for some $q \geq \frac{n}{2}$. Let $\mu \in A_1(\mathbb{R}^n) \cap B_{e_0}$, where $e_0 = \frac{(s_1-1)q'}{s_1-q'}$ with $s_1 \in (q', s)$ and $s > q'$. Assume $\mu^{1-p} \in A_{\frac{p}{q'}}$. For every $b \in BMO_\mu$, there exists a constant $C_p$ such that
\[
q' < p < \infty, \quad \|[b, T_4]f\|_{L^p(\mu^{1-p})} \leq C_p\|b\|_{BMO_\mu}\|f\|_{L^p(\mu)} \tag{15}
\]
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and
\[ 1 < p < q, \quad \| [b, T_A^s] f \|_{L^p(\mu^{1-p})} \leq C_p \| b \|_{BMO(\mu)} \| f \|_{L^p(\mu)}. \]

**Remark 2** The results in [3] are special cases of Theorem 4, 5, 6 and Corollary 3, when \( \mu \equiv 1 \).

## 2 Proofs of Lemmas and Theorems

Proofs of Theorem 1, 2 and 3 mainly depend on the following Proposition 1. We first discuss the problem for general operator \( T f(x) = \int K(x, y) f(y) dy \). Later, we will specialize to \( T_j(j = 1, 2, 3) \). Although employed to prove Theorem 1, 2 and 3, Proposition 1 has its independent significance, for the weight function \( \mu \) in Proposition 1 is weaker than those in Theorem 1, 2 and 3.

**Proposition 1** Let \( m > 1, m' < p < r < \infty \) and \( \frac{1}{r} = \frac{1}{p} - \frac{\beta}{n} \) for \( 0 < \beta < 1 \). Suppose \( K \) satisfies \( H(m) \). Assume \( \mu \in A_1(\mathbb{R}^n) \cap B_{c_0} \), where \( \epsilon_0 = \frac{(s_1 - 1)m'}{s_1 - m'} \) for some \( s_1 \in (m', s) \) and \( m' < s < \frac{n}{\beta} \). Moreover, \( T \) is bounded on \( L^q(\mu^{1-q}) \) for every \( q \in (m', \infty) \). For every \( b \in Lip_{\beta, \mu} \), then \([b, T]\) is bounded from \( L^p(\mu) \) to \( L^r(\mu^{1-r}) \) for every \( p \in (m', \infty) \), and
\[
\| [b, T] f \|_{L^r(\mu^{1-r})} \leq C_p \| b \|_{Lip_{\beta, \mu}} \| f \|_{L^p(\mu)}. \tag{12}
\]

We adopt the Störmberg’s idea. Proposition 1 follows immediately from the following Lemma 1 and a theorem of Fefferman-Stein on sharp function.

**Lemma 1** Let \( m > 1, \frac{1}{r} = \frac{1}{p} - \frac{\beta}{n} \) for \( 0 < \beta < 1 \) and \( s \in (m', \frac{n}{\beta}) \). Assume \( K \) satisfies \( H(m) \) and \( \mu \in A_1(\mathbb{R}^n) \cap B_{c_0} \), where \( \epsilon_0 = \frac{(s_1 - 1)m'}{s_1 - m'} \) for some \( s_1 \in (m', s) \). Suppose \( T \) is bounded on \( L^q(\mu^{1-q}) \) for every \( q \in (m', \infty) \). Then there exists constant \( C_s > 0 \) such that \( \forall f \in L^1_{loc}, b \in Lip_{\beta, \mu} \),
\[
M^\#(b, T f)(x) \leq C_s \mu(x) \| b \|_{Lip_{\beta, \mu}} \left\{ M_{\beta, \mu, s}(T f)(x) + M_{\beta, \mu, s}(f)(x) \right\} \tag{13}
\]
holds.

**Proof.** Fix \( s \in (m', \frac{n}{\beta}) \), \( f \in L^1_{loc} \), \( x \in \mathbb{R}^n \), and fix a ball \( B = B(x_0, l) \) with \( x \in B \). We only need to control \( J = \frac{1}{|B|} \int_B \| [b, T] f(y) - ([b, T] f)_B \| dy \) by the right side of (13). Let \( f = f_1 + f_2 \), where \( f_1 = f \chi_{32B}, f_2 = f - f_1 \). Then \([b, T] f = [b - b_B, T] f = (b - b_B) (T f - T(b - b_B) f_1 - T(b - b_B) f_2) = A_1 f + A_2 f + A_3 f \), and we get
\[
J \leq \frac{1}{|B|} \int_B |A_1 f(y) - A_1 f_B| dy
\]

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+ \frac{1}{|B|} \int_B |A_2 f(y) - A_2 f_B| dy + \frac{1}{|B|} \int_B |A_3 f(y) - A_3 f_B| dy
= J_1 + J_2 + J_3.

For $J_1$, by Hölder's inequality and (7), we can obtain

$$J_1 \leq C \frac{1}{|B|} \int_B |A_1 f(y)| dy$$
$$= C \frac{1}{|B|} \int_B |(b - b_B) T f(y)| dy$$
$$\leq C \frac{1}{|B|^{\frac{1}{n}}} \left( \frac{1}{|B|} \int_B |b - b_B|^s \mu(y)^{1-s'} dy \right)^{\frac{1}{2}} \left( \frac{1}{|B|^{1 - \frac{s'}{n}}} \int_B |T f(y)| \mu(y) dy \right)^{\frac{1}{2}}$$
$$\leq C \frac{\mu(B)}{|B|} \frac{1}{\mu(B)^{\frac{1}{n}}} \left( \frac{1}{\mu(B)} \int_B |b - b_B|^s \mu(y)^{1-s'} dy \right)^{\frac{1}{2}} \left( \mu(B)^{1 - \frac{s'}{n}} \int_B |T f(y)| \mu(y) dy \right)^{\frac{1}{2}}$$
$$\leq C \mu(x) \|b\|_{L^{p,\beta,\mu}} M_{\beta,\mu}(T f)(x).$$

Considering $J_2$, we fix $s_1$ such that $s > s_1 > m'$, and let $s_2 = \frac{ns_1}{s - s_1}$. Since weighted $L^{s_1}(\mu^{1-s_1})$ boundedness of $T$, we can obtain

$$J_2 \leq C \frac{1}{|B|} \int_B |T((b - b_B) f_1)(y)| dy$$
$$\leq C \left( \frac{1}{|B|} \int_B |T((b - b_B) f_1)(y)|^{s_1} \mu(y)^{1-s_1} dy \right)^{\frac{1}{s_1}} \left( \frac{1}{|B|} \int_B \mu(y) dy \right)^{\frac{s_1-1}{s_1}}$$
$$\leq C \left( \frac{1}{|B|} \int_B |(b(y) - b_B)| f_1(y)|\mu(y)^{1-s_1} dy \right)^{\frac{1}{s_1}} \left( \mu(B) \frac{1}{|B|} \right)^{\frac{s_1-1}{s_1}}$$
$$\leq C \left( \frac{\mu(B)}{|B|} \frac{1}{\mu(B)^{\frac{1}{n}}} \left( \frac{1}{\mu(32B)^{\frac{1}{n}}} \int_{32B} |b(y)| \mu(y) dy \right)^{\frac{1}{2}} \right)^{\frac{1}{s_1}}$$
$$\leq C \mu(x) \|b\|_{L^{p,\beta,\mu}} M_{\beta,\mu}(f)(x).$$

To estimate $J_3$, we set $c_B = \int_{|z - x_0| > 32t} K(x_0, z)(b(z) - b_B) f(z) dz$, then

$$J_3 \leq C \frac{1}{|B|} \int_B |A_3 f(y) - c_B| dy$$
$$\leq C \frac{1}{|B|} \int_B \int_{|z - x_0| > 32t} |K(y, z) - K(x_0, z)|(b(z) - b_B) f(z) dz dy$$
$$= C \frac{1}{|B|} \int_B \sum_{k=5}^{\infty} \int_{2^k \leq |z - x_0| < 2^{k+1}} \left| [(K(y, z) - K(x_0, z))(b(z) - b_B) f(z)] dz \right| dy$$

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\begin{align*}
\leq & \frac{C}{|B|} \int_B \sum_{k=5}^{\infty} \left\{ \int_{2^k |z-x_0| < 2^{k+1}} |K(y,z) - K(x_0,z)|^m dz \right\}^{\frac{1}{m}} |(2^k l)|^{\frac{s}{m}} k \\
& \times \left( \frac{1}{(2^k l)^{|s\beta - m|}} \int_{|z-x_0| < 2^{k+1}} |(b(z) - b_B) f(z)|^m dz \right)^{\frac{1}{m}} dy \\
\leq & \sup_{k \geq 5} \frac{1}{k} \left\{ \int_{|z-x_0| < 2^{k+1}} |(b(z) - b_B) f(z)|^m dz \right\}^{\frac{1}{m}} \\
\leq & C \sup_{k \geq 5} \frac{1}{k} \left\{ \int_{|z-x_0| < 2^{k+1}} |(b(z) - b_{2^{k+1}} + b_{2^{k+1}} - b_B) f(z)|^m dz \right\}^{\frac{1}{m}} \\
& \times \left\{ \int_{|z-x_0| < 2^{k+1}} |(b_{2^{k+1}} - b_B) f(z)|^m dz \right\}^{\frac{1}{m}} \\
\leq & C \sup_{k \geq 5} \frac{1}{k} \left( E_1 + E_2 \right).
\end{align*}

For $E_1$, fix $s_1 \in (m', s)$, then $\frac{s_1}{m'} > 1, \left( \frac{s_1}{m'} \right)' = \frac{s_1}{s_1 - m'}; \frac{s}{m'} > 1, \left( \frac{s}{m'} \right)' = \frac{s}{s-m'}$. Applying Hölder’s inequality twice, by $\mu \in B_{\epsilon_0}, \epsilon_0 = \frac{(s_1-1)m'}{s_1-m'}$, we can get

\begin{align*}
E_1 &= \left\{ \frac{C}{|2^{k+1} B|} \int_{2^{k+1} B} |(b(z) - b_{2^{k+1}}) \mu(z) \right\}^{(s_1-1)} \left\{ f(z) \mu(z) \right\}^{s_1} |m' \mu(z)^{m'(1 - \frac{1}{s_1})} dz \right\}^{\frac{1}{m'}} \\
& \leq \left\{ \frac{C}{|2^{k+1} B|} \int_{2^{k+1} B} |(b(z) - b_{2^{k+1}}) \mu(z) \right\}^{(s_1-1)} \left\{ f(z) \mu(z) \right\}^{s_1} | \mu(z) \right\}^{s_1} dz \right\}^{\frac{1}{m'}} \\
& \times \left\{ \int_{2^{k+1} B} |(b(z) - b_{2^{k+1}}) \mu(z) |^{s_1-1} | \mu(z) |^{s_1} dz \right\}^{\frac{1}{m'}} \\
& \leq \left\{ \frac{C}{|2^{k+1} B|} \int_{2^{k+1} B} |f(z) |^{s_1} | \mu(z) |^{s_1} dz \right\}^{\frac{1}{m'}} \\
& \times \left\{ \frac{1}{|2^{k+1} B|} \int_{2^{k+1} B} |(b(z) - b_{2^{k+1}}) \mu(z) |^{s_1} | \mu(z) |^{s_1} dz \right\}^{\frac{1}{m'}} \\
& \leq \left\{ \frac{1}{|2^{k+1} B|} \int_{2^{k+1} B} |f(z) |^{s_1} | \mu(z) |^{s_1} dz \right\}^{\frac{1}{m'}} \\
& \times \left\{ \frac{1}{|2^{k+1} B|} \int_{2^{k+1} B} |(b(z) - b_{2^{k+1}}) \mu(z) |^{s_1} | \mu(z) |^{s_1} dz \right\}^{\frac{1}{m'}} \\
& \leq C \mu(x) \|b\|_{Lip_{\beta, \mu, M_{\beta, \mu, s}}}(f)(x).
\end{align*}

For $E_2$, since $\frac{s}{m'} > 1, \left( \frac{s}{m'} \right)' = \frac{s}{s-m'}$, applying Hölder’s inequality, $\mu \in A_1(\mathbb{R}^n)$ and the fact $|b_{2^{k+1}} - b_B| \leq C(k+1) \mu(2^{k+1} B) \frac{m'}{m} \|b\|_{Lip_{\beta, \mu}}$, we can get

\begin{align*}
E_2 &= |b_{2^{k+1}} - b_B| \left\{ \frac{1}{|2^{k+1} B|} \int_{2^{k+1} B} |f(z) |^{s_1} | \mu(z) |^{s_1} \mu(z) |^{s_1} dz \right\}^{\frac{1}{m'}} \\
& \leq C \mu(x) \|b\|_{Lip_{\beta, \mu, M_{\beta, \mu, s}}}(f)(x).
\end{align*}
\[ \leq |b_{2^{k+1}B} - b_B| \left\{ \frac{1}{|2^{k+1}B|} \int_{2^{k+1}B} |f(z)|^s \mu(z) dz \right\}^{\frac{1}{s}} \times \left\{ \frac{1}{|2^{k+1}B|} \int_{2^{k+1}B} \mu(z) \left( -\frac{m'}{\lambda} + \frac{\nu}{\lambda} \right)' dz \right\}^{\frac{1}{\nu'}} \leq C(k + 1) \left( \frac{\mu(2^{k+1}B)}{|2^{k+1}B|} \right)^{\frac{1}{s}} \|b\|_{Lip_{\beta,p}} \left\{ \frac{1}{\mu(2^{k+1}B)} \right\}^{\frac{1}{s}} \times \left\{ \int_{2^{k+1}B} \mu(z) \left( -\frac{1}{\lambda^2} \right) dz \right\}^{\frac{1}{\lambda^2 - 1}} \leq C(k + 1) \left( \frac{\mu(2^{k+1}B)}{|2^{k+1}B|} \right)^{\frac{1}{s}} \|b\|_{Lip_{\beta,p}} M_{\beta,p,s}(f)(x) \left( \frac{|2^{k+1}B|}{\mu(2^{k+1}B)} \right)^{\frac{1}{s}} \leq C(k + 1) \mu(x) \|b\|_{Lip_{\beta,p}} M_{\beta,p,s}(f)(x). \]

Combining estimates of \( E_1 \) and \( E_2 \), we obtain

\[
J_3 = C \sup_{k \geq 5} \frac{1}{k} \left( E_1 + E_2 \right) \leq C \sup_{k \geq 5} \frac{k + 1}{k} \mu(x) \|b\|_{Lip_{\beta,p}} M_{\beta,p,s}(f)(x) \leq C \mu(x) \|b\|_{Lip_{\beta,p}} M_{\beta,p,s}(f)(x). \]

This completes the proof of Lemma 1.

**Proof of Proposition 1.** From Lemma 1, since \( \mu \in A_1(\mathbb{R}^n) \) implies \( \mu^{1-r} \in A_r(\mathbb{R}^n) \), we can get

\[
\|b, T \|_{L^r(\mu^{1-r})} \leq \|M^\#(b, T \|f(x)\|_{L^r(\mu^{1-r})} \leq C \|b\|_{Lip_{\beta,p}} \left( \|M_{\beta,p,s}(Tf)\|_{L^r(\mu)} + \|M_{\beta,p,s}(f)\|_{L^r(\mu)} \right) \leq C \|b\|_{Lip_{\beta,p}} \|f\|_{L^p(\mu)}. \]

**Proof of Theorem 1** Firstly, by (1) of Theorem 1.7 in [8],

\[
|T_1 f(x)| \leq C M \left( |f^{\#}| \right)^{\frac{2}{p'}}(x) \quad (f \in C_0^\infty(\mathbb{R}^n)), \]

where \( \frac{1}{q} + \frac{1}{q'} = 1 \). \( \forall p > q' \), since \( \mu^{1-p} \in A_{q'}, \) it is known that \( M \) is bounded on \( L^p(\mu^{1-p}) \). Hence \( T_1 \) is bounded on \( L^p(\mu^{1-p}) \). Since Z.Guo, P.Li and L.Peng have shown that the kernel \( K_1 \) of \( T_1 \) satisfies \( H(q) \) in [1], from Proposition 1, we can get Theorem 1.

**Proof of Theorem 2** We can get \( L^p(\mu^{1-p}) \) boundedness of \( T_2 \) from Proof of Theorem 5.10 in [6]. For the sake of complete, we give the sketch as follow. It follows from [6] that

\[
T_2^* f(x) \leq C \int_{\mathbb{R}^n} \frac{V(x)^{1/2} f(y) dy}{\left( 1 + m(y, V) |x - y| \right)^{k|x - y|^{n-1}}}, \]

where \( V(x) \) is the kernel of \( T_2 \).
and
\[ |T_2 f(x)| \leq C \left\{ M(|f|^{(2q)'}) (x) \right\}^{\frac{1}{2q'}}, \]
\[ \forall p > (2q)', \text{ since } \mu^{1-p} \in A_{\frac{p}{(2q)'}, } \text{ it is known that } M \text{ is bounded on } L^{\frac{p}{2q'}} (\mu^{1-p}). \]

Hence
\[ \|T_2 f(x)\|_{L^p(\mu^{1-p})} \leq C\| f\|_{L^p(\mu^{1-p})}, \quad \text{for} \quad (2q)' < p < \infty. \]

Since Z.Guo, P.Li and L.Peng have shown that the kernel \( K_2 \) of \( T_2 \) satisfies \( H((2q)') \) in [1], from Proposition 1, we can get Theorem 2.

**Proof of Theorem 3** \( L^p(\mu^{1-p}) \) boundedness of \( T_3 \) can be obtained by the following estimate
\[ |T_3 f(x)| \leq C \left\{ M(|f|^{p_0'}) (x) \right\}^{\frac{1}{p_0'}} + 2\mathcal{T} f(x), \]
where \( q < n, \frac{1}{p_0'} = \frac{1}{q} - \frac{1}{n} \) and \( \mathcal{T} = \nabla(-\Delta)^{-\frac{1}{2}} \) is a Calderón-Zygmund operator, which can be found in [6].

Because \( p_0' < p < \infty \), and \( \mu^{1-p} \in A_{\frac{p}{p_0}, } \subset A_p \), we have
\[ \|T_3 f\|_{L^p(\mu^{1-p})} \leq C \left\{ \int_{\mathbb{R}^n} \left\{ M(|f|^{p_0'}) (x) \right\}^{\frac{p}{p_0'}} \mu^{1-p} dx \right\}^p + C\| f\|_{L^p(\mu^{1-p})} \]
\[ \leq C \left\{ \int_{\mathbb{R}^n} \left\| (2q)' (x) \right\|^{\frac{p}{2q'}} \mu^{1-p} dx \right\}^p + C\| f\|_{L^p(\mu^{1-p})} \]
\[ = C\| f\|_{L^p(\mu^{1-p})}. \]

Since Z.Guo, P.Li and L.Peng have shown that the kernel \( K_3 \) of \( T_3 \) satisfies \( H(p_0') \) in [1], from Proposition 1, we can get Theorem 3.

**Proof of Corollary 1**. Because
\[ T_4 = (-\Delta + V)^{-1}\nabla^2 = (-\Delta + V)^{-1}(-\Delta)(-\Delta)^{-1}\nabla^2 \]
\[ = (I - (-\Delta + V)^{-1}V)\nabla^2 (-\Delta) = (I - T_1)\nabla^2 (-\Delta), \]
we get
\[ [b, T_4] = [b, I - T_1]\nabla^2 (-\Delta) - (I - T_1)[b, \nabla^2 (-\Delta)], \quad (14) \]

So, according to \( (L^p(\mu), L^r(\mu^{1-r})) \) boundedness of \([b, I - T_1]|(Theorem 1) \) and \([b, \nabla^2 (-\Delta)]|\text{see [4]}, \]
\( L^r(\mu^{1-r}) \) boundedness of \( I - T_1 \), and \( L^p(\mu) \) boundedness of \( \frac{\nabla^2}{(-\Delta)} \text{ (see [7])} \), we can get
\[ \|[b, T_4] f\|_{L^r(\mu^{1-r})} \leq \|[b, I - T_1]\nabla^2 (-\Delta)f\|_{L^r(\mu^{1-r})} + \|(I - T_1)[b, \nabla^2 (-\Delta)] f\|_{L^r(\mu^{1-r})} \]
\[ \leq C\|b\|_{L^{p_0}(\overline{\mu})} \|\nabla^2 (-\Delta) f\|_{L^p(\mu)} + C\|b, \nabla^2 (-\Delta) f\|_{L^r(\mu^{1-r})} \]
\[ \leq C\|b\|_{L^{p_0}(\overline{\mu})} \| f\|_{L^p(\mu)}. \]
References


