Weighted Boundedness of Commutators of Riesz Transforms Associated with Schrödinger Operator*

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Abstract. In this paper, we consider Schrödinger operator $-\Delta+V(x)$ on $\mathbb{R}^n(n\geq 3)$, where $V(x)$ is non-zero, non-negative, and belongs to reverse Hölder classes B_q for some $q\geq\frac{n}{2}$. Let $T_1=(-\Delta+V)^{-1}V, T_2=(-\Delta+V)^{-\frac{1}{2}}V^{\frac{1}{2}}, T_3=(-\Delta+V)^{-\frac{1}{2}}\nabla, T_4=(-\Delta+V)^{-1}\nabla^2$. We show that the commutators $[b,T_j](j=1,2,3,4)$ are bounded operators from $L^p(\mu)$ to $L^r(\mu^{1-r})$, when $b\in Lip_{\beta,\mu}(0<\beta<1)$, for some weight function μ. The weighted boundedness of commutators of weighted BMO functions and $T_j(j=1,2,3,4)$ also are obtained similarly.

Key words: Schrödinger operator; Riesz transforms; Reverse Hölder classes; Weighted Lipschitz function; Commutator

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1 Introduction

Let $P=-\Delta+V(x)$ be the Schrödinger differential operator on $\mathbb{R}^n, n\geq 3$. We assume that $V(x)$ is a non-zero, non-negative potential, and belongs to reverse Hölder classes B_q for some $q\geq\frac{n}{2}$. Let $T_j(j=1,2,3,4)$ be the Riesz transforms associated to Schrödinger operators, namely ,$T_1=(-\Delta+V)^{-1}V, T_2=(-\Delta+V)^{-\frac{1}{2}}V^{\frac{1}{2}}, T_3=(-\Delta+V)^{-\frac{1}{2}}\nabla$ and $T_4=(-\Delta+V)^{-1}\nabla^2$. J. Zhong (see[1])proved that if V is a non-negative polynomial, $\nabla^2(-\Delta+V)^{-1}, \nabla(-\Delta+V)^{-\frac{1}{2}}$and $\nabla(-\Delta+V)^{-1}\nabla$ are Calderón-Zygmund operators. Z. Shen generalized these results(see[2]). He

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proved that $\nabla(-\Delta+V)^{-\frac{1}{2}}, (-\Delta+V)^{-\frac{1}{2}}\nabla$and $\nabla(-\Delta+V)^{-1}\nabla$ are Calderón-Zygmund operators, if V belongs to the reverse Hölder class B_n, which includes non-negative polynomials and allows some non-smooth potentials. Moreover, Z.Shen also shown L^p boundedness for T_1, T_2, T_3 and T_4 when $V \in B_{\frac{n}{2}}$. Recently, Z.Guo, P.Li and L. Peng (see [3]) studied L^p boundedness of commutators $[b, T_j] = bT_j - T_j b(j = 1, 2, 3, 4)$, when $b \in BMO(\mathbb{R}^n)$. Canqing Tang and Bolin Ma(see [4]) proved L^p boundedness of commutators $[b, T_j](j = 1, 2, 3, 4)$, when b is a Lipschitz function. In this paper we will generalize results in [3]and[4] to weighted case.

Definition 1 A non-negative locally L^q integrable function $V(x)$ on $\mathbb{R}^n$ is said to belong to $B_q(1 < q < \infty)$, if there exists $C > 0$ such that the reverse Hölder inequality

$$\Big(\frac{1}{|B|}\int_B V(y)^q dy\Big)^{\frac{1}{q}} \leq C\Big(\frac{1}{|B|}\int_B V(y)dy\Big), \tag{1}$$

holds for every ball B in $\mathbb{R}^n$.

By Hölder inequality, it is easy to see that $B_{q_1} \subset B_{q_2}(q_1 > q_2 > 1)$. One remarkable feature about the B_q class is that, if $V \in B_q$ for some $q > 1$, then there exists $\epsilon > 0$, which depends only on n and the constant C in (1), such that $V \in B_{q+\epsilon}$(see [5]). It is also well known that, if $V \in B_q(q > 1)$, then $V(x)dx$ is a doubling measure, namely for any $r > 0, x \in \mathbb{R}^n$, there exists a constant $C_0 > 0$ such that

$$\int_{B(x,2r)} V(y)dy \leq C_0 \int_{B(x,r)} V(y)dy. \tag{2}$$

It was proved that if $V \in B_n$, then T_3 is a Calderón-Zygmund operator(see[2]). According to M.Paluszyński's classical result in [6], if T is a Calderón-Zygmund operator, $b \in Lip_\beta(0 < \beta < 1)$ if and only if the commutator $[b, T]$ is bounded from L^p to L^q, where $1 < p < q < \infty, \frac{1}{q} = \frac{1}{p} - \frac{\beta}{n}$, then the commutator $[b, T_3]$ and $[b, T_3^*]$ are bounded from L^p to L^q, where $1 < p < q < \infty, \frac{1}{q} = \frac{1}{p} - \frac{\beta}{n}$. Bei Hu, Jiajun Gu in [7] shown that for $\mu \in A_1$, $b \in Lip_{\beta,\mu}$ if and only if the commutator $[b, T]$ is bounded from $L^p(\mu)$ to $L^q(\mu^{1-q})$, where $1 < p < q < \infty, 0 < \beta < 1$, and $\frac{1}{q} = \frac{1}{p} - \frac{\beta}{n}$. However, in [3], the authors shown these kernels had no smoothness of C-Z kernel due to $V \in B_q$, for some $q > \frac{n}{2}$ and they discovered that the kernels have some other kind of smoothness.

Definition 2[3] $K(x, y)$ is said to satisfy $H(m)$ for some $m \geq 1$, if there exists a constant $C \geq 0$ such that, $\forall l > 0, x, x_0 \in \mathbb{R}^n$ with $|x - x_0| \leq l$, then

$$\sum_{k=5}^{\infty} k(2^k l)^{\frac{n}{m'}} \Big(\int_{2^k l \leq |y-x_0| < 2^{k+1} l} |K(x, y) - K(x_0, y)|^m dy\Big) < C, \tag{3}$$

where $\frac{1}{m} + \frac{1}{m'} = 1$.

It is easy to see that if $K(x,y)$ is a C-Z kernels ,then $K(x,y)$ satisfies $H(m)$, for $m \geq 1$. In [3], the authors proved that these kernels of $T_j(j=1,2,3)$ satisfy $H(m)$, m in different ranges respectively.

We give some notations. A non-negative function μ defined on $\mathbb{R}^n$ is called weight if it is locally integral. A weighted μ is said to belong to Muckenhoupt class $A_p(\mathbb{R}^n)$ for $1<p<\infty$, if there exists a constant C such that

$$\frac{1}{|B|}\int_B \mu(x)dx\Big(\frac{1}{|B|}\int_B \mu(x)^{\frac{-1}{p-1}}dx\Big)^{(p-1)} \leq C \tag{4}$$

holds for every ball $B \subset \mathbb{R}^n$; The class $A_1(\mathbb{R}^n)$ is defined replacing the above inequality by

$$\frac{1}{|B|}\int_B \mu(x)dx \leq C\mu(x), \qquad a.e. x \in \mathbb{R}^n \tag{5}$$

for every ball $B(\ni x) \subset \mathbb{R}^n$(see[8]). It was well known that $A_{p_1} \subset A_{p_2}$, for $1 \leq p_1 \leq p_2 < \infty$ and $\mu \in A_1$ implies $\mu^{1-p} \in A_p(1<p<\infty)$. We note that $A_\infty = \bigcup_{p\geq 1} A_p$.

Definition 3 We will say that a locally integrable function $f(x)$ belongs to the weighted $Lip^p_{\beta,\mu}$ for $1 \leq p < \infty, 0<\beta<1$ and $\mu \in A_\infty(\mathbb{R}^n)$, that is

$$\sup_B \frac{1}{\mu(B)^{\frac{\beta}{n}}}\Big[\frac{1}{\mu(B)}\int_B |f(y)-f_B|^p\mu(y)^{1-p}dy\Big]^{\frac{1}{p}} \leq C < \infty, \tag{6}$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$.

Modulo constants, the Banach space of such function is denoted by $Lip^p_{\beta,\mu}$. The smallest bound C satisfying conditions above is then taken to the norm of f in these spaces, and is denoted by $\|f\|_{Lip^p_{\beta,\mu}}$. Put $Lip_{\beta,\mu} = Lip^1_{\beta,\mu}$. Obviously, for the case $\mu = 1$, the $Lip_{\beta,\mu}$ is the classical Lip_β space.

Two basic facts about $Lip^p_{\beta,\mu}$ may be in order(see [9]).

$$For \quad \mu \in A_1, \qquad \|f\|_{Lip^p_{\beta,\mu}} \sim \|f\|_{Lip_{\beta,\mu}}, \qquad \forall p>1. \tag{7}$$

We use 2^kB to denote the ball with the same center as B but with 2^k times radius of it, then

$$|f_{2^kB} - f_B| \leq Ck\mu(B)^{\frac{\beta}{n}}\|f\|_{Lip_{\beta,\mu}}. \tag{8}$$

We also denote the fractional weighted maximum function of function $f(x)$ by

$$M_{\beta,\mu,r}f(x) = \sup_{x\in B}\Big(\frac{1}{\mu(B)^{1-\frac{r\beta}{n}}}\int_B |f(y)|^r\mu(y)dy\Big)^{\frac{1}{r}}, \qquad 0<r<\frac{n}{\beta}. \tag{9}$$

Our main theorems are as follows.

Theorem 1 Suppose $V \in B_q$ for some $q \geq \frac{n}{2}, \frac{1}{r} = \frac{1}{p} - \frac{\beta}{n}$ for $0<\beta<1$ and $q' < p < r < \infty$. Let $\mu \in A_1(\mathbb{R}^n)\bigcap B_{\epsilon_0}$, where $\epsilon_0 = \frac{(s_1-1)q'}{s_1-q'}$ for some $s_1 \in (q',s)$ and

$s \in (q', \frac{n}{\beta})$. Assume $\mu^{1-p} \in A_{\frac{p}{q'}}$. For every $b \in Lip_{\beta,\mu}$, there exists a constant C_p such that

$$\|[b, T_1]\|_{L^r(\mu^{1-r})} \leq C_p \|b\|_{Lip_{\beta,\mu}} \|f\|_{L^p(\mu)}.$$

Theorem 2 Suppose $V \in B_q$ for some $q \geq \frac{n}{2}, \frac{1}{r} = \frac{1}{p} - \frac{\beta}{n}$ for $0 < \beta < 1$ and $(2q)' < p < r < \infty$. Let $\mu \in A_1(\mathbb{R}^n) \bigcap B_{\epsilon_0}$, where $\epsilon_0 = \frac{(s_1-1)(2q)'}{s_1-(2q)'}$ for some $s_1 \in ((2q)', s)$ and $s \in ((2q)', \frac{n}{\beta})$. Assume $\mu^{1-p} \in A_{\frac{p}{(2q)'}}$. For every $b \in Lip_{\beta,\mu}$, there exists a constant C_p such that

$$\|[b, T_2]\|_{L^r(\mu^{1-r})} \leq C_p \|b\|_{Lip_{\beta,\mu}} \|f\|_{L^p(\mu)}.$$

Theorem 3 Suppose $V \in B_q$ for some $q \geq \frac{n}{2}, \frac{1}{r} = \frac{1}{p} - \frac{\beta}{n}$ for $0 < \beta < 1, \frac{1}{p_0} = \frac{1}{q} - \frac{1}{n}$ and $p_0' < p < r < \infty$. Let $\mu \in A_1(\mathbb{R}^n) \bigcap B_{\epsilon_0}$,where $\epsilon_0 = \frac{(s_1-1)p_0'}{s_1-p_0'}$ for some $s_1 \in (p_0', s)$ and $s \in (p_0', \frac{n}{\beta})$. Assume $\mu^{1-p} \in A_{\frac{p}{p_0'}}$. For every $b \in Lip_{\beta,\mu}$, then

$$\|[b, T_3]\|_{L^r(\mu^{1-r})} \leq C_p \|b\|_{Lip_{\beta,\mu}} \|f\|_{L^p(\mu)}$$

Let T^* denote adjoint operator of T. We know that $T_1^* = V(-\Delta + V)^{-1}, T_2^* = V^{\frac{1}{2}}(-\Delta + V)^{-\frac{1}{2}}, T_3^* = -\nabla(-\Delta + V)^{-\frac{1}{2}}$. By duality, under the same assumptions as in Theorem 1, 2 and 3, we can easily obtain that

$$\|[b, T_1^*]\|_{L^r(\mu^{1-r})} \leq C_p \|b\|_{Lip_{\beta,\mu}} \|f\|_{L^p(\mu)}, \qquad 1 < p < q,$$

$$\|[b, T_2^*]\|_{L^r(\mu^{1-r})} \leq C_p \|b\|_{Lip_{\beta,\mu}} \|f\|_{L^p(\mu)}, \qquad 1 < p < 2q,$$

$$\|[b, T_3^*]\|_{L^r(\mu^{1-r})} \leq C_p \|b\|_{Lip_{\beta,\mu}} \|f\|_{L^p(\mu)}, \qquad 1 < p < p_0,$$

where $\frac{1}{p_0} = \frac{1}{q} - \frac{1}{n}$.

We know that $T_4^* = \nabla^2(-\Delta + V)^{-1}$. By Theorem 1, we conclude that

Corollary 1 Suppose $V \in B_q$ for some $q \geq \frac{n}{2}, \frac{1}{r} = \frac{1}{p} - \frac{\beta}{n}, 0 < \beta < 1$ and $1 < p < r < \infty$. Let $\mu \in A_1(\mathbb{R}^n) \bigcap B_{\epsilon_0}$, where $\epsilon_0 = \frac{(s_1-1)q'}{s_1-q'}$ for some $s_1 \in (q', s)$ and $s \in (q', \frac{n}{\beta})$. Assume $\mu^{1-p} \in A_{\frac{p}{q'}}$. For every $b \in Lip_{\beta,\mu}$, we have

$$q' < p < \infty, \qquad \|[b, T_4]f\|_{L^r(\mu^{1-r})} \leq C_p \|b\|_{Lip_{\beta,\mu}} \|f\|_{L^p(\mu)}$$

and

$$1 < p < q, \qquad \|[b, T_4^*]f\|_{L^r(\mu^{1-r})} \leq C_p \|b\|_{Lip_{\beta,\mu}} \|f\|_{L^p(\mu)}.$$

Remark 1 The results in [4] are the special cases of Theorem 1, 2 3 and Corollary 1, when $\mu \equiv 1$.

We also consider the commutators of Riesz Transforms Associated to Schrödinger Operator and weighted BMO function.

Definition 4 Let $1 \le p < \infty$, and $\mu \in A_\infty(\mathbb{R}^n)$. We say a local integrable function $f(x)$ belongs to BMO_μ^p, if there exists a constant $C > 0$ such that

$$\sup_B \Big[\frac{1}{\mu(B)} \int_B |f(y) - f_B|^p \mu(y)^{1-p} dy\Big]^{\frac{1}{p}} \le C < \infty,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$.

Modulo constants, the Banach space of such function is denoted by BMO_μ^p. The smallest bound C satisfying conditions above is then taken to the norm of f in these spaces, and is denoted by $\|f\|_{BMO_\mu^p}$. Put $BMO_\mu = BMO_\mu^1$. Obviously, for the case $\mu = 1$, the BMO_μ is the classical BMO space.

Two basic facts about BMO_μ^p may be in order(see [9]).

$$\mu \in A_1, \qquad \|f\|_{BMO_\mu^p} \sim \|f\|_{BMO_\mu}, \qquad \forall p > 1. \tag{10}$$

We use $2^k B$ to denote the ball with the same center as B but with 2^k times radius of it, then

$$|f_{2^k B} - f_B| \le C(k+1)\|f\|_{BMO_\mu}. \tag{11}$$

Applying the same proving routine of theorems and corollaries above, we can easily obtain the following theorems and corollary.

Theorem 4 Suppose $V \in B_q$ for some $q \ge \frac{n}{2}, q' \le p < \infty$. Let $\mu \in A_1(\mathbb{R}^n) \bigcap B_{\epsilon_0}$, where $\epsilon_0 = \frac{(s_1-1)q'}{s_1-q'}$ with some $s_1 \in (q', s)$ and $s > q'$. Assume $\mu^{1-p} \in A_{\frac{p}{q'}}$. For every $b \in BMO_\mu$, there exists a constant C_p such that

$$\|[b, T_1]f\|_{L^p(\mu^{1-p})} \le C_p \|b\|_{BMO_\mu} \|f\|_{L^p(\mu)}.$$

Theorem 5 Suppose $V \in B_q$ for some $q \ge \frac{n}{2}, (2q)' < p < \infty$. Let $\mu \in A_1(\mathbb{R}^n) \bigcap B_{\epsilon_0}$, where $\epsilon_0 = \frac{(s_1-1)(2q)'}{s_1-(2q)'}$ with some $s_1 \in ((2q)', s)$ and $s > (2q)'$. Assume $\mu^{1-p} \in A_{\frac{p}{(2q)'}}$. For every $b \in BMO_\mu$, there exists a constant C_p such that

$$\|[b, T_2]f\|_{L^p(\mu^{1-p})} \le C_p \|b\|_{BMO_\mu} \|f\|_{L^p(\mu)}.$$

Theorem 6 Suppose $V \in B_q$ for some $q \ge \frac{n}{2}, \frac{1}{p_0} = \frac{1}{q} - \frac{1}{n}$ and $p_0' < p < r < \infty$. Let $\mu \in A_1(\mathbb{R}^n) \bigcap B_{\epsilon_0}$, where $\epsilon_0 = \frac{(s_1-1)p_0'}{s_1-p_0'}$ with some $s_1 \in (p_0', s)$ and $s > p_0'$. Assume $\mu^{1-p} \in A_{\frac{p}{p_0'}}$. For every $b \in BMO_\mu$, there exists a constant C_p such that

$$\|[b, T_3]\|_{L^r(\mu^{1-r})} \le C_p \|b\|_{BMO_\mu} \|f\|_{L^p(\mu)}.$$

Corollary 3 Suppose $V \in B_q$ for some $q \ge \frac{n}{2}$. Let $\mu \in A_1(\mathbb{R}^n) \bigcap B_{\epsilon_0}$, where $\epsilon_0 = \frac{(s_1-1)q'}{s_1-q'}$ with $s_1 \in (q', s)$ and $s > q'$. Assume $\mu^{1-p} \in A_{\frac{p}{q'}}$. For every $b \in BMO_\mu$, there exists a constant C_p such that

$$q' < p < \infty, \qquad \|[b, T_4]f\|_{L^p(\mu^{1-p})} \le C_p \|b\|_{BMO_\mu} \|f\|_{L^p(\mu)}$$

and

$$1 < p < q, \qquad \|[b, T_4^*]f\|_{L^p(\mu^{1-p})} \leq C_p \|b\|_{BMO_\mu} \|f\|_{L^p(\mu)}.$$

Remark 2 The results in [3] are special cases of Theorem 4, 5, 6 and Corollary 3, when $\mu \equiv 1$.

2 Proofs of Lemmas and Theorems

Proofs of Theorem 1, 2 and 3 mainly depend on the following Proposition 1. We first discuss the problem for general operator $Tf(x) = \int K(x,y)f(y)dy$. Later, we will specialize to $T_j (j = 1, 2, 3)$. Although employed to prove Theorem 1,2 and 3, Proposition 1 has its independent significance, for the weight function μ in Proposition 1 is weaker than those in Theorem 1,2 and 3.

Proposition 1 Let $m > 1, m' < p < r < \infty$ and $\frac{1}{r} = \frac{1}{p} - \frac{\beta}{n}$ for $0 < \beta < 1$. Suppose K satisfies $H(m)$. Assume $\mu \in A_1(\mathbb{R}^n) \bigcap B_{\epsilon_0}$, where $\epsilon_0 = \frac{(s_1-1)m'}{s_1-m'}$ for some $s_1 \in (m', s)$ and $m' < s < \frac{n}{\beta}$. Moreover, T is bounded on $L^q(\mu^{1-q})$ for every $q \in (m', \infty)$. For every $b \in Lip_{\beta,\mu}$, then $[b, T]$ is bounded from $L^p(\mu)$ to $L^r(\mu^{1-r})$ for every $p \in (m', \infty)$, and

$$||[b, T]f||_{L^r(\mu^{1-r})} \leq C_p ||b||_{Lip_{\beta,\mu}} ||f||_{L^p(\mu)}. \tag{12}$$

We adopt the Störmberg's idea. Proposition 1 follows immediately from the following Lemma 1 and a theorem of Fefferman-Stein on sharp function.

Lemma 1 Let $m > 1, \dfrac{1}{r} = \dfrac{1}{p} - \dfrac{\beta}{n}$ for $0 < \beta < 1$ and $s \in (m', \frac{n}{\beta})$. Assume K satisfies $H(m)$ and $\mu \in A_1(\mathbb{R}^n) \bigcap B_{\epsilon_0}$, where $\epsilon_0 = \frac{(s_1-1)m'}{s_1-m'}$ for some $s_1 \in (m', s)$. Suppose T is bounded on $L^q(\mu^{1-q})$ for every $q \in (m', \infty)$. Then there exists constant $C_s > 0$ such that $\forall f \in L^1_{loc}, b \in Lip_{\beta,\mu}$,

$$M^{\#}([b, T]f)(x) \leq C_s \mu(x) \|b\|_{Lip_\beta,\mu} \Big\{ M_{\beta,\mu,s}(Tf)(x) + M_{\beta,\mu,s}(f)(x) \Big\} \tag{13}$$

holds.

Proof. Fix $s \in (m', \frac{n}{\beta}), f \in L^1_{loc}, x \in \mathbb{R}^n$, and fix a ball $B = B(x_0, l)$ with $x \in B$. We only need to control $J = \frac{1}{|B|} \int_B |[b, T]f(y) - ([b, T]f)_B| dy$ by the right side of (13). Let $f = f_1 + f_2$, where $f_1 = f\chi_{32B}, f_2 = f - f_1$. Then $[b, T]f = [b - b_B, T]f = (b - b_B)Tf - T(b - b_B)f_1 - T(b - b_B)f_2 = A_1 f + A_2 f + A_3 f$, and we get

$$J \leq \frac{1}{|B|} \int_B |A_1 f(y) - A_1 f_B| dy$$

$$
\begin{aligned}
&+\frac{1}{|B|}\int_B |A_2f(y)-A_2f_B|dy+\frac{1}{|B|}\int_B |A_3f(y)-A_3f_B|dy\\
=\ & J_1+J_2+J_3.
\end{aligned}
$$

For J_1, by Hölder's inequality and (7), we can obtain

$$
\begin{aligned}
J_1 &\le C\frac{1}{|B|}\int_B |A_1f(y)|dy\\
&= C\frac{1}{|B|}\int_B |(b-b_B)Tf(y)|dy\\
&\le C\frac{1}{|B|^{\frac{\beta}{n}}}\Big(\frac{1}{|B|}\int_B |b-b_B|^{s'}\mu(y)^{1-s'}dy\Big)^{\frac{1}{s'}}\Big(\frac{1}{|B|^{1-\frac{s\beta}{n}}}\int_B |Tf(y)|^s\mu(y)dy\Big)^{\frac{1}{s}}\\
&\le C\frac{\mu(B)}{|B|}\frac{1}{\mu(B)^{\frac{\beta}{n}}}\Big(\frac{1}{\mu(B)}\int_B |b-b_B|^{s'}\mu(y)^{1-s'}dy\Big)^{\frac{1}{s'}}\\
&\quad\times\Big(\frac{1}{\mu(B)^{1-\frac{s\beta}{n}}}\int_B |Tf(y)|^s\mu(y)dy\Big)^{\frac{1}{s}}\\
&\le C\mu(x)\|b\|_{Lip_{\beta,\mu}}M_{\beta,\mu,s}(Tf)(x).
\end{aligned}
$$

Considering J_2, we fix s_1 such that $s>s_1>m'$, and let $s_2=\frac{ss_1}{s-s_1}$. Since weighted $L^{s_1}(\mu^{1-s_1})$ boundedness of T, we can obtain

$$
\begin{aligned}
J_2 &\le C\frac{1}{|B|}\int_B |T((b-b_B)f_1)(y)|dy\\
&\le C\Big(\frac{1}{|B|}\int_B |T((b-b_B)f_1)(y)|^{s_1}\mu(y)^{1-s_1}dy\Big)^{\frac{1}{s_1}}\Big(\frac{1}{|B|}\int_B \mu(y)dy\Big)^{\frac{s_1-1}{s_1}}\\
&\le C\Big(\frac{1}{|B|}\int_B |(b(y)-b_B)f_1(y)|^{s_1}\mu(y)^{1-s_1}dy\Big)^{\frac{1}{s_1}}\Big(\frac{\mu(B)}{|B|}\Big)^{\frac{s_1-1}{s_1}}\\
&\le C\Big(\frac{\mu(B)}{|B|}\Big)^{\frac{s_1-1}{s_1}+\frac{1}{s_2}+\frac{1}{s}}\frac{1}{\mu(32B)^{\frac{\beta}{n}}}\Big(\frac{1}{\mu(32B)}\int_{32B} |b(y)-b_B|^{s_2}\mu(y)^{1-s_2}dy\Big)^{\frac{1}{s_2}}\\
&\quad\times\Big(\frac{1}{\mu(32B)^{1-\frac{s\beta}{n}}}\int_{32B} |f(y)|^s\mu(y)dy\Big)^{\frac{1}{s}}\\
&\le C\mu(x)\|b\|_{Lip_{\beta,\mu}}M_{\beta,\mu,s}(f)(x).
\end{aligned}
$$

To estimate J_3, we set $c_B=\int_{|z-x_0|>32l}K(x_0,z)(b(z)-b_B)f(z)dz$, then

$$
\begin{aligned}
J_3 &\le C\frac{1}{|B|}\int_B |A_3f(y)-c_B|dy\\
&\le C\frac{1}{|B|}\int_B |\int_{|z-x_0|>32l}[K(y,z)-K(x_0,z)](b(z)-b_B)f(z)dz|dy\\
&= C\frac{1}{|B|}\int_B \sum_{k=5}^{\infty}\int_{2^kl\le|z-x_0|<2^{k+1}l}|[K(y,z)-K(x_0,z)](b(z)-b_B)f(z)|dzdy
\end{aligned}
$$

$$
\begin{aligned}
&\leq C\frac{1}{|B|}\int_B\sum_{k=5}^{\infty}\Big\{\int_{2^kl\leq|z-x_0|<2^{k+1}l}|K(y,z)-K(x_0,z)|^m dz\Big\}^{\frac{1}{m}}(2^kl)^{\frac{n}{m'}}k\\
&\quad\times\frac{1}{(2^kl)^{\frac{n}{m'}}k}\Big\{\int_{2^kl\leq|z-x_0|<2^{k+1}l}|(b(z)-b_B)f(z)|^{m'}dz\Big\}^{\frac{1}{m'}}dy\\
&\leq C\sup_{k\geq5}\frac{1}{k}\Big\{\frac{1}{(2^kl)^n}\int_{2^kl\leq|z-x_0|<2^{k+1}l}|(b(z)-b_B)f(z)|^{m'}dz\Big\}^{\frac{1}{m'}}\\
&\leq C\sup_{k\geq5}\frac{1}{k}\Big\{\frac{1}{(2^kl)^n}\int_{|z-x_0|<2^{k+1}l}|(b(z)-b_{2^{k+1}B}+b_{2^{k+1}B}-b_B)f(z)|^{m'}dz\Big\}^{\frac{1}{m'}}\\
&\leq C\sup_{k\geq5}\frac{1}{k}\Big(\Big\{\frac{1}{(2^kl)^n}\int_{|z-x_0|<2^{k+1}l}|(b(z)-b_{2^{k+1}B})f(z)|^{m'}dz\Big\}^{\frac{1}{m'}}\\
&\quad+\Big\{\frac{1}{(2^kl)^n}\int_{|z-x_0|<2^{k+1}l}|(b_{2^{k+1}B}-b_B)f(z)|^{m'}dz\Big\}^{\frac{1}{m'}}\Big)\\
&\leq C\sup_{k\geq5}\frac{1}{k}\Big(E_1+E_2\Big).
\end{aligned}
$$

For E_1, fix $s_1\in(m',s)$, then $\frac{s_1}{m'}>1$, $\Big(\frac{s_1}{m'}\Big)'=\frac{s_1}{s_1-m'}$; $\frac{s}{m'}>1$, $\Big(\frac{s}{m'}\Big)'=\frac{s}{s-m'}$. Applying Hölder's inequality twice, by $\mu\in B_{\epsilon_0}$, $\epsilon_0=\frac{(s_1-1)m'}{s_1-m'}$, we can get

$$
\begin{aligned}
E_1 &= \Big\{\frac{C}{|2^{k+1}B|}\int_{2^{k+1}B}|(b(z)-b_{2^{k+1}B})\mu(z)^{\frac{1}{s_1}-\frac{1}{s}-1}f(z)\mu(z)^{\frac{1}{s}}|^{m'}\mu(z)^{m'(1-\frac{1}{s_1})}dz\Big\}^{\frac{1}{m'}}\\
&\leq \Big\{\frac{C}{|2^{k+1}B|}\int_{2^{k+1}B}|(b(z)-b_{2^{k+1}B})\mu(z)^{\frac{1}{s_1}-\frac{1}{s}-1}f(z)\mu(z)^{\frac{1}{s}}|^{s_1}dz\Big\}^{\frac{1}{s_1}}\\
&\quad\times\Big\{\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B}\mu(z)^{\frac{(s_1-1)m'}{s_1-m'}}dz\Big\}^{\frac{s_1-m'}{s_1m'}}\\
&\leq \Big\{\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B}(|b(z)-b_{2^{k+1}B}|\mu(z)^{\frac{1}{s_1}-\frac{1}{s}-1})^{s_1(\frac{s}{s_1})'}dz\Big\}^{\frac{1}{s_1}\frac{1}{(\frac{s}{s_1})'}}\\
&\quad\times\Big\{\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B}|f(z)|^s\mu(z)dz\Big\}^{\frac{1}{s}}\Big\{\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B}\mu(z)dz\Big\}^{\frac{s_1-1}{s_1}}\\
&\leq \Big\{\frac{\mu(2^{k+1}B)}{|2^{k+1}B|}\Big\}^{\frac{s-s_1}{ss_1}}\Big\{\frac{1}{\mu(2^{k+1}B)}\int_{2^{k+1}B}|b(z)-b_{2^{k+1}B}|^{\frac{ss_1}{s-s_1}}\mu(z)^{1-\frac{ss_1}{s-s_1}}dz\Big\}^{\frac{s-s_1}{ss_1}}\\
&\quad\times\Big\{\frac{\mu(2^{k+1}B)}{|2^{k+1}B|}\Big\}^{\frac{1}{s}}\Big\{\frac{1}{\mu(2^{k+1}B)}\int_{2^{k+1}B}|f(z)|^s\mu(z)dz\Big\}^{\frac{1}{s}}\Big\{\frac{\mu(2^{k+1}B)}{|2^{k+1}B|}\Big\}^{\frac{s_1-1}{s_1}}\\
&\leq C\mu(x)\|b\|_{Lip_{\beta,\mu}}M_{\beta,\mu,s}(f)(x).
\end{aligned}
$$

For E_2, since $\frac{s}{m'}>1$, $\Big(\frac{s}{m'}\Big)'=\frac{s}{s-m'}$, applying Hölder's inequality , $\mu\in A_1(\mathbb{R}^n)$ and the fact $|b_{2^{k+1}B}-b_B|\leq C(k+1)\mu(2^{k+1}B)^{\frac{\beta}{n}}\|b\|_{Lip_{\beta,\mu}}$, we can get

$$
E_2 = |b_{2^{k+1}B}-b_B|\Big\{\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B}|f(z)|^{m'}\mu(z)^{\frac{m'}{s}}\mu(z)^{-\frac{m'}{s}}dz\Big\}^{\frac{1}{m'}}
$$

$$
\begin{aligned}
&\leq |b_{2^{k+1}B}-b_B|\Big\{\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B}|f(z)|^s\mu(z)dz\Big\}^{\frac{1}{s}}\\
&\quad\times\Big\{\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B}\mu(z)^{(-\frac{m'}{s})(\frac{s}{m'})'}dz\Big\}^{\frac{1}{m'}(\frac{s}{m'})'}\\
&\leq C(k+1)\Big(\frac{\mu(2^{k+1}B)}{|2^{k+1}B|}\Big)^{\frac{1}{s}}\|b\|_{Lip_{\beta,\mu}}\Big\{\frac{1}{|\mu(2^{k+1}B)^{1-\frac{s\beta}{n}}}\int_{2^{k+1}B}|f(z)|^s\mu(z)dz\Big\}^{\frac{1}{s}}\\
&\quad\times\Big\{\frac{1}{|2^{k+1}B|}\Big\{\int_{2^{k+1}B}\mu(z)^{-\frac{1}{\frac{s}{m'}-1}}dz\Big\}^{\frac{s}{m'}-1}\Big\}^{\frac{1}{s}}\\
&\leq C(k+1)\Big(\frac{\mu(2^{k+1}B)}{|2^{k+1}B|}\Big)^{\frac{1}{s}}\|b\|_{Lip_{\beta,\mu}}M_{\beta,\mu,s}(f)(x)\Big(\frac{|2^{k+1}B|}{\mu(2^{k+1}B)}\Big)^{\frac{1}{s}}\\
&\leq C(k+1)\mu(x)\|b\|_{Lip_{\beta,\mu}}M_{\beta,\mu,s}(f)(x).
\end{aligned}
$$

Combining estimates of E_1 and E_2, we obtain

$$
\begin{aligned}
J_3 &= C\sup_{k\geq 5}\frac{1}{k}\Big(E_1+E_2\Big)\\
&\leq C\sup_{k\geq 5}\frac{k+1}{k}\mu(x)\|b\|_{Lip_{\beta,\mu}}M_{\beta,\mu,s}(f)(x)\\
&\leq C\mu(x)\|b\|_{Lip_{\beta,\mu}}M_{\beta,\mu,s}(f)(x).
\end{aligned}
$$

This completes the proof of Lemma 1.

Proof of Proposition 1. From Lemma 1, since $\mu\in A_1(\mathbb{R}^n)$ implies $\mu^{1-r}\in A_r(\mathbb{R}^n)$, we can get

$$
\begin{aligned}
\|[b,T]f(x)\|_{L^r(\mu^{1-r})} &\leq \|M^{\#}\Big([b,T]f(x)\Big)\|_{L^r(\mu^{1-r})}\\
&\leq C\|b\|_{Lip_{\beta,\mu}}\Big(\|M_{\beta,\mu,s}(Tf)\|_{L^r(\mu)}+\|M_{\beta,\mu,s}(f)\|_{L^r(\mu)}\Big)\\
&\leq C\|b\|_{Lip_{\beta,\mu}}\|f\|_{L^p(\mu)}.
\end{aligned}
$$

Proof of Theorem 1 Firstly, by (1) of Theorem 1.7 in [8],

$$
|T_1f(x)|\leq CM((|f|^{q'})^{\frac{1}{q'}}(x)\qquad\qquad (f\in C_0^\infty(\mathbb{R}^n)),
$$

where $\frac{1}{q}+\frac{1}{q'}=1$. $\forall p>q'$, since $\mu^{1-p}\in A_{\frac{p}{q'}}$, it is known that M is bounded on $L^{\frac{p}{q'}}(\mu^{1-p})$. Hence T_1 is bounded on $L^p(\mu^{1-p})$. Since Z.Guo, P.Li and L.Peng have shown that the kernel K_1 of T_1 satisfies $H(q)$ in [1], from Proposition 1, we can get Theorem 1.

Proof of Theorem 2 We can get $L^p(\mu^{1-p})$ boundedness of T_2 from Proof of Theorem 5.10 in [6]. For the sake of complete, we give the sketch as follow. It follows from [6] that

$$
T_2^*f(x)\leq C\int_{\mathbb{R}^n}\frac{V(x)^{1/2}f(y)dy}{\{1+m(y,V)|x-y|\}^k|x-y|^{n-1}},
$$

and

$$|T_2 f(x)| \le C\{M(|f|^{(2q)'})(x)\}^{\frac{1}{(2q)'}},$$

$\forall p > (2q)'$, since $\mu^{1-p} \in A_{\frac{p}{(2q)'}}$, it is known that M is bounded on $L^{\frac{p}{(2q)'}}(\mu^{1-p})$. Hence

$$\|T_2 f(x)\|_{L^p(\mu^{1-p})} \le C\|f\|_{L^p(\mu^{1-p})}, \quad for \qquad (2q)' < p < \infty.$$

Since Z.Guo, P.Li and L.Peng have shown that the kernel K_2 of T_2 satisfies $H((2q)')$ in [1], from Proposition 1, we can get Theorem 2.

Proof of Theorem 3 $L^p(\mu^{1-p})$ boundedness of T_3 can be obtained by the following estimate

$$|T_3 f(x)| \le C\{M(|f|^{p_0'})(x)\}^{\frac{1}{p_0'}} + 2\mathcal{T} f(x),$$

where $q < n$, $\frac{1}{p_0} = \frac{1}{q} - \frac{1}{n}$ and $\mathcal{T} = \nabla(-\Delta)^{-\frac{1}{2}}$ is a Calderón-Zygmund operator, which can be found in [6].

Because $p_0' < p < \infty$, and $\mu^{1-p} \in A_{\frac{p}{p_0'}} \subset A_p$, we have

$$\begin{aligned}
\|T_3 f\|_{L^p(\mu^{1-p})} &\le C\{\int_{\mathbb{R}^n} \{M(|f|^{p_0'})(x)\}^{\frac{p}{p_0'}} \mu^{1-p} dx\}^p + C\|\mathcal{T} f\|_{L^p(\mu^{1-p})} \\
&\le C\{\int_{\mathbb{R}^n} \{|f|^{(2q)'}(x)\}^{\frac{p}{(2q)'}} \mu^{1-p} dx\}^p + C\|f\|_{L^p(\mu^{1-p})} \\
&= C\|f\|_{L^p(\mu^{1-p})}.
\end{aligned}$$

Since Z.Guo, P.Li and L.Peng have shown that the kernel K_3 of T_3 satisfies $H(p_0')$ in [1], from Proposition 1, we can get Theorem 3.

Proof of Corollary 1. Because

$$\begin{aligned}
T_4 &= (-\Delta + V)^{-1}\nabla^2 = (-\Delta + V)^{-1}(-\Delta)(-\Delta)^{-1}\nabla^2 \\
&= (I - (-\Delta + V)^{-1}V)\frac{\nabla^2}{(-\Delta)} = (I - T_1)\frac{\nabla^2}{(-\Delta)},
\end{aligned}$$

we get

$$[b, T_4] = [b, I - T_1]\frac{\nabla^2}{(-\Delta)} - (I - T_1)[b, \frac{\nabla^2}{(-\Delta)}]. \tag{14}$$

So, according to $(L^p(\mu), L^r(\mu^{1-r}))$ boundedness of $[b, I - T_1]$(Theorem 1) and $[b, \frac{\nabla^2}{(-\Delta)}]$(see [4]), $L^r(\mu^{1-r})$ boundedness of $I - T_1$, and $L^p(\mu)$ boundedness of $\frac{\nabla^2}{(-\Delta)}$(see [7]), we can get

$$\begin{aligned}
\|[b, T_4]f\|_{L^r(\mu^{1-r})} &\le \|[b, I - T_1]\frac{\nabla^2}{(-\Delta)} f\|_{L^r(\mu^{1-r})} + \|(I - T_1)[b, \frac{\nabla^2}{(-\Delta)}]f\|_{L^r(\mu^{1-r})} \\
&\le C\|b\|_{Lip_\beta}\|\frac{\nabla^2}{(-\Delta)} f\|_{L^p(\mu)} + C\|[b, \frac{\nabla^2}{(-\Delta)}]f\|_{L^r(\mu^{1-r})} \\
&\le C\|b\|_{Lip_\beta}\|f\|_{L^p(\mu)}.
\end{aligned}$$

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