An Expression of Arbitrary Positive Integer Powers for One Type of Tridiagonal Matrices

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Abstract—In the [7] and [8] papers, J. Rimas researched the tridiagonal matrices with elements 1, 0, 0, . . . , 1 in principal and 1, 1, 1, . . . , 1 in neighbouring diagonals, and gave its high power of expression. In this new paper, we will derive the general expression of the high power of tridiagonal matrices with elements 1, 0, 0, . . . , 0 in principal and 1, 1, 1, . . . , 1 in neighbouring diagonals. This new paper will use the polynomial of the recurrence relation and trigonometric nature, discuss the tridiagonal matrix, calculate high-order matrix into the low-order matrix of the matrix and launch its high power of expression.

I. INTRODUCTION AND PRELIMINARIES

Tridiagonal are used in different areas of science and engineering (solution of ordinary difference systems and partial difference [1], [2], telecommunication system analysis [3], image processing and coding [4], and so on). A problem of computing the arbitrary positive integer powers of such matrices often arises in [5], [6], [7], [8]. Recently, there have been several papers on computing the positive integer powers of various kinds of square matrices by Professors Rimas who gave the general expression of the $l$th power for one type of symmetric tridiagonal matrices. In this paper, we will derive the general expression of the $l$th ($l \in \mathbb{N}$) power for another type of tridiagonal matrix.

In the [8] paper Rimas considered the tridiagonal matrix of $n$th order as the following type:

$$B = \begin{pmatrix}
1 & 1 & & \\
1 & 0 & 1 & \\
& \ldots & \ldots & \\
1 & 0 & 1 & \\
& & & 1 & 1
\end{pmatrix}$$

(1)

and gave its $l$th ($l \in \mathbb{N}$) power of the expression, as the following

$$B^l(i,j) = \sum_{k=1}^{n} t_k \lambda_k^l T_{2k-1} \left( \frac{\lambda_k}{2} \right) T_{2k-1} \left( \frac{\lambda_k}{2} \right)$$

(2)

here

$$t_k = \begin{cases}
\frac{2}{n} & k = 1, \ldots, n-1 \\
\frac{1}{n} & k = n
\end{cases}$$

(3)

$\lambda_k$ and $T_k(k = 1, \ldots, n)$ are the eigenvalues and eigenvectors of the matrix $B$.

This new paper will derive the another type of the tridiagonal matrix:

$$A = \begin{pmatrix}
1 & 1 & & \\
1 & 0 & 1 & \\
& \ldots & \ldots & \\
1 & 0 & 1 & \\
& & & 1 & 0
\end{pmatrix}$$

(4)

but we can’t obtain its eigenvalues, eigenvectors and high power general expression according to the conclusion of the matrix $B$.

II. GENERAL EXPRESSION OF THE $l$TH POWER

In this section we should firstly calculate the eigenvalues and eigenvectors of the tridiagonal matrix $A$ of the form (4). Since $A$ is a real symmetric matrix, the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of $A$ are real. Then we can choose the corresponding eigenvectors of $A$, say $T_1, T_2, \ldots, T_n$ to be real [10]. Thus, $A = T \Lambda T^{-1}$, where $\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, $T = (T_1, T_2, \ldots, T_n)$, then $A^l = T \Lambda^l T^{-1}$. In order to get $A^l$ we need calculate the eigenvalues and associated eigenvectors of $A$.

Theorem 1. Let $A$ of the form (4), then $A$ has eigenvalues

$$\lambda_j = -2 \cos \frac{2j}{2n+1} \pi, j = 1, 2, \ldots, n$$

(5)
and associated eigenvectors

\[ T_j = \begin{pmatrix}
(-1)^{n+1} \sin \frac{n^2 j \pi}{2n+1} \\
(-1)^n \sin \frac{(n-1)2 j \pi}{2n+1} \\
\vdots \\
(-1)^2 \sin \frac{2 j \pi}{2n+1}
\end{pmatrix}, j = 1, 2, \ldots, n. \quad (6)
\]

**Proof.** The eigenvalues of \( A \) are defined by the characteristic equation

\[ |A - \lambda I_n| = 0. \quad (7)
\]

where \( I_n \) is the identity matrix of the \( n \)th order. Let us denote

\[
D_n(\alpha) = \begin{pmatrix}
\alpha+1 & 1 & 0 \\
1 & \alpha & 1 \\
& \ddots & \ddots \\
0 & 1 & \alpha
\end{pmatrix}, \quad \Delta_n(\alpha) = \begin{pmatrix}
\alpha & 1 & 0 \\
1 & \alpha & 1 \\
& \ddots & \ddots \\
0 & 1 & \alpha
\end{pmatrix}, \quad (8)
\]

where \( \alpha \in \mathbb{R} \). Then

\[ |A - \lambda I_n| = D_n(-\lambda). \quad (10)
\]

From (8) (9) follows:

\[
D_n = (\alpha + 1)\Delta_{n-1} - \Delta_{n-2} \quad \text{and} \quad \Delta_n = \alpha \Delta_{n-1} - \Delta_{n-2}(\Delta_2 = \alpha^2 - 1, \Delta_1 = \alpha, \Delta_0 = 1),
\]

here \( D_n = D_n(\alpha), \Delta_n = \Delta_n(\alpha) \).

Solving difference equation (7), we obtain

\[ \Delta_n(\alpha) = U_n(\frac{\alpha}{2}), D_n(\alpha) = U_n(\frac{\alpha}{2}) + U_n-1(\frac{\alpha}{2}). \]

Here \( U_n(x) \) are the \( n \)th degree Chebyshev polynomials of the second kind respectively [9]:

\[ U_n(x) = \frac{\sin(n+1) \arccos x}{\sin \arccos x}, -1 \leq x \leq 1. \]

All the roots of the polynomial \( U_n(x) \) are included in the interval \([-1, 1]\) and can be found using the relation [9]:

\[ x_{nj} = \cos \frac{j \pi}{n+1}, j = 1, 2, \ldots, n. \quad (11)
\]

Taking (10) (11) into account we find the roots of the characteristic equation (7) (the eigenvalues of the matrix \( A \)):

\[ \lambda_j = -2 \cos \frac{2 j \pi}{2n+1}, j = 1, 2, \ldots, n. \]

Since all the eigenvalues \( \lambda_j(j = 1, 2, \ldots, n) \) are simple, for each eigenvalue \( \lambda_j \) corresponds single Jordan cell \( J_1(\lambda_j) \) in the matrix \( J \). Taking this into account we can get the Jordan’s form of the matrix \( A \):

\[ J = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n). \]

Consider the relation \( J = T^{-1}AT(\lambda A = T \lambda J) \), here \( A \) is the \( n \)th order \( (n \in N, n \geq 2) \) matrix, \( J \) is the Jordan’s form of \( A, T \) is the transforming matrix. Denoting \( j \)th column of \( T \) by \( T_j \), we have \( T = (T_1, T_2, \ldots, T_n) \) and \( (AT_1, AT_2, \ldots, AT_n) = (\lambda_1 T_1, \lambda_2 T_2, \ldots, \lambda_n T_n) \). The latter expression gives

\[ AT_j = \lambda_j T_j, j = 1, 2, \ldots, n. \quad (12)
\]

Solving the set of systems (12), we find the associated eigenvectors of matrix \( A \):

\[
T_j = \begin{pmatrix}
(-1)^{n+1} \sin \frac{n^2 j \pi}{2n+1} \\
(-1)^n \sin \frac{(n-1)2 j \pi}{2n+1} \\
\vdots \\
(-1)^2 \sin \frac{2 j \pi}{2n+1}
\end{pmatrix}, j = 1, 2, \ldots, n.
\]

With above preparation, we can now prove our main results.

**Theorem 2.** Let the \( n \times n \) matrix \( A \) be defined by (4) and \( l \) be a positive integer, then the entry of the \( l \)th power of \( A \) is given by

\[
A^l(i, j) = \frac{4}{2n+1} \left(-1\right)^{i+j} \sum_{k=1}^{n} \lambda_k^l \sin \frac{2k(n-i+1)}{2n+1} \pi \cdot \sin \frac{2k(n-j+1)}{2n+1} \pi, \quad (13)
\]

where \( \lambda_j = -2 \cos \frac{2 j \pi}{2n+1}, i, j = 1, 2, \ldots, n. \)

**Proof.** Let \( A = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n), T = (T_1, T_2, \ldots, T_n) \), here \( \lambda_j, T_j \) are defined by (5) and (6) respectively. Then \( A = T \Lambda T^{-1} \) and

\[ T^T T = \frac{(2n+1)}{4} I_n \]

where \( I_n \) is the identity matrix of the \( n \)th order.

Thus, \( T^{-1} = \frac{4}{2n+1} T^T \). Then

\[ A^l = T \Lambda^l T^{-1} = \frac{4}{2n+1} T \Lambda^l T^T. \quad (14)
\]

Applying (6) (14) we can obtain the following formula:

\[
A^l(i, j) = \frac{4}{2n+1} \left(-1\right)^{i+j} \sum_{k=1}^{n} \lambda_k^l \sin \frac{2k(n-i+1)}{2n+1} \pi \cdot \sin \frac{2k(n-j+1)}{2n+1} \pi.
\]
Therefore, for \( l \in \mathbb{N}, i, j = 1, 2, \ldots, n \) we can get the general expression (13) of the entry of \( A \).

Since \( A \) is symmetric, \( A^t \) is also symmetric. We only need to compute half of the entries of \( A \). That is, we only need to compute \( A(i, j) \) for \( j = i, i + 1, \ldots, n \) when \( i = 1, 2, \ldots, n \).

With above the conclusion, we can further consider the \( n \)th order tridiagonal matrix with the following form:

\[
C = \begin{pmatrix}
1 & 1 & 1 & \cdots & 1 \\
1 & 0 & 1 & \cdots & 1 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
1 & 0 & 1 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 1
\end{pmatrix}.
\] (15)

**Theorem 3.** Let the \( n \times n \) matrix \( C \) be defined by (15) and \( l \) be a positive integer, then the entry of the \( l \)th power of \( C \) is given by

\[
C^l(i, j) = \begin{cases} 
\frac{4}{2n-1} (-1)^{i+j-\delta_{jn}} \sum_{k=1}^{n-1} \lambda_k^{l-\delta_{jn}} \cdot \sin \frac{2k(n-i)}{2n-1} \pi \sin \frac{2k(n-j)}{2n-1} \pi & i \neq n \\
0 & i = n
\end{cases}
\] (16)

where

\[
\delta_{jn} = \begin{cases} 
1, & j = n \\
0, & j \neq n
\end{cases}
\]

and \( \lambda_j = -2 \cos \frac{2j\pi}{2n+1}, j = 1, 2, \ldots, n \).

**Proof.** Partition \( C \) as

\[
C = \begin{pmatrix}
A & e_n \\
0 & 0
\end{pmatrix},
\] (18)

where \( e_n \) is a unit vector in \( \mathbb{R}^{n-1} \), \( A \in \mathbb{R}^{(n-1) \times (n-1)} \) has form (4). It can be verified by induction that

\[
C^l = \begin{pmatrix}
A^l & A^{l-1}e_n \\
0 & 0
\end{pmatrix}.
\] (19)

It follows from (13) and (19) that for \( i = 1, \ldots, n-1; j = 1, \ldots, n-1 \)

\[
C^l(i, j) = A^l(i, j) = \frac{4}{2n-1} (-1)^{i+j} \sum_{k=1}^{n-1} \lambda_k^l \cdot \sin \frac{2k(n-i)}{2n-1} \pi \sin \frac{2k(n-j)}{2n-1} \pi,
\] (20)

and for \( j = n; i = 1, \ldots, n-1 \)

\[
C^l(i, j) = A^{l-1}(i, j-1) = \frac{4}{2n-1} (-1)^{i+j-1} \sum_{k=1}^{n-1} \lambda_k^{l-1} \cdot \sin \frac{2k(n-i)}{2n-1} \pi \sin \frac{2k(n-j+1)}{2n-1} \pi.
\] (21)

Combination of (19)-(21) gives (16).

**III. NUMERICAL EXAMPLES**

From (5) (6) (13) we can derive the arbitrary powers of the \( n \)th order matrix (4). The algorithms were implemented in MATLAB7.11. For example , when \( n = 5 \), we get from (5) (6) that

\[
A = \text{diag}(-1.6825, 0.8308, 0.2846, 1.3097, 1.9190),
\]

\[
T = \begin{pmatrix}
0.2817 & -0.5406 & 0.7557 & -0.9096 & 0.9898 \\
-0.7557 & 0.9898 & -0.5406 & -0.2817 & 0.9096 \\
0.9898 & -0.2817 & -0.9096 & 0.5406 & 0.7557 \\
-0.9096 & -0.7557 & 0.2817 & 0.9898 & 0.5406 \\
0.5406 & 0.9096 & 0.9898 & 0.7557 & 0.2817
\end{pmatrix}.
\]

Then we get from (13) that

\[
A^6(i, j) = (-1)^{i+j} \frac{4}{11} \sum_{k=1}^{5} \lambda_k^6 T_{ik} T_{jk}, i, j = 1, 2, \ldots, 5.
\]

Since \( A^6 \) is also symmetric, we only need to compute half of the entries of \( A^6 \) as follows:

\[
A^6(1, 1) = (-1)^{1+1} \frac{4}{11} (\lambda_1^6 T_{11}^2 + \lambda_2^6 T_{12}^2 + \lambda_3^6 T_{13}^2 + \lambda_4^6 T_{14}^2 + \lambda_5^6 T_{15}^2) = 20.0000,
\]

\[
A^6(1, 2) = (-1)^{1+2} \frac{4}{11} (\lambda_1^6 T_{11} T_{21} + \lambda_2^6 T_{12} T_{22} + \lambda_3^6 T_{13} T_{23} + \lambda_4^6 T_{14} T_{24} + \lambda_5^6 T_{15} T_{25}) = 15.0000,
\]

\[
A^6(1, 3) = (-1)^{1+3} \frac{4}{11} (\lambda_1^6 T_{11} T_{31} + \lambda_2^6 T_{12} T_{32} + \lambda_3^6 T_{13} T_{33} + \lambda_4^6 T_{14} T_{34} + \lambda_5^6 T_{15} T_{35}) = 15.0000,
\]

\[
A^6(1, 4) = (-1)^{1+4} \frac{4}{11} (\lambda_1^6 T_{11} T_{41} + \lambda_2^6 T_{12} T_{42} + \lambda_3^6 T_{13} T_{43} + \lambda_4^6 T_{14} T_{44} + \lambda_5^6 T_{15} T_{45}) = 6.0000,
\]

\[
A^6(1, 5) = (-1)^{1+5} \frac{4}{11} (\lambda_1^6 T_{11} T_{51} + \lambda_2^6 T_{12} T_{52} + \lambda_3^6 T_{13} T_{53} + \lambda_4^6 T_{14} T_{54} + \lambda_5^6 T_{15} T_{55}) = 5.0000,
\]

\[
A^6(2, 2) = (-1)^{2+2} \frac{4}{11} (\lambda_1^6 T_{21}^2 + \lambda_2^6 T_{22}^2 + \lambda_3^6 T_{23}^2 + \lambda_4^6 T_{24}^2 + \lambda_5^6 T_{25}^2) = 20.0000,
\]

\[
A^6(2, 3) = (-1)^{2+3} \frac{4}{11} (\lambda_1^6 T_{21} T_{31} + \lambda_2^6 T_{22} T_{32} + \lambda_3^6 T_{23} T_{33} + \lambda_4^6 T_{24} T_{34} + \lambda_5^6 T_{25} T_{35}) = 6.0000,
\]

\[
A^6(2, 4) = (-1)^{2+4} \frac{4}{11} (\lambda_1^6 T_{21} T_{41} + \lambda_2^6 T_{22} T_{42} + \lambda_3^6 T_{23} T_{43} + \lambda_4^6 T_{24} T_{44} + \lambda_5^6 T_{25} T_{45}) = 14.0000,
\]

\[
A^6(2, 5) = (-1)^{2+5} \frac{4}{11} (\lambda_1^6 T_{21} T_{51} + \lambda_2^6 T_{22} T_{52} + \lambda_3^6 T_{23} T_{53} + \lambda_4^6 T_{24} T_{54} + \lambda_5^6 T_{25} T_{55}) = 1.0000,
\]

By the same method, we can get the remaining results:

\[
A^6(3, 3) = 19.0000,
\]

\[
A^6(3, 4) = 1.0000,
\]

\[
A^6(3, 5) = 9.0000,
\]

\[
A^6(4, 4) = 14.0000,
\]

\[
A^6(4, 5) = 0,
\]

\[
A^6(5, 5) = 5.0000.
\]
Since all entries of $A^6$ are integers and the above results calculated by MATLAB7.11 are decimal fractions with 5 significant figures, then we obtain

$$A^6 = \begin{pmatrix} 20 & 15 & 15 & 6 & 5 \\ 15 & 20 & 6 & 14 & 1 \\ 15 & 6 & 19 & 1 & 9 \\ 6 & 14 & 1 & 14 & 0 \\ 5 & 1 & 9 & 0 & 5 \end{pmatrix}.$$ 

Now for $C$ of order 6, we have

$$C^6 = \begin{pmatrix} 20 & 15 & 15 & 6 & 5 & 1 \\ 15 & 20 & 6 & 14 & 1 & 4 \\ 15 & 6 & 19 & 1 & 9 & 0 \\ 6 & 14 & 1 & 14 & 0 & 5 \\ 5 & 1 & 9 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$ 

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